

# $N = 2$ Supersymmetry and Bailey Pairs

Alexander Berkovich <sup>1</sup>

*Physikalisches Institut der  
Rheinischen Friedrich-Wilhelms Universität Bonn  
Nussallee 12  
D-53115 Bonn, Germany*

and

Barry M. McCoy <sup>2</sup> and Anne Schilling<sup>3</sup>

*Institute for Theoretical Physics  
State University of New York  
Stony Brook, NY 11794-3840*

---

<sup>1</sup> berkovic@axpib.physik.uni-bonn.de

<sup>2</sup> mccoy@max.physics.sunysb.edu

<sup>3</sup> anne@insti.physics.sunysb.edu

## Abstract

We demonstrate that the Bailey pair formulation of Rogers-Ramanujan identities unifies the calculations of the characters of  $N = 1$  and  $N = 2$  supersymmetric conformal field theories with the counterpart theory with no supersymmetry. We illustrate this construction for the  $M(3, 4)$  (Ising) model where the Bailey pairs have been given by Slater. We then present the general unitary case. We demonstrate that the model  $M(p, p + 1)$  is derived from  $M(p - 1, p)$  by a Bailey renormalization flow and conclude by obtaining the  $N = 1$  model  $SM(p, p + 2)$  and the unitary  $N = 2$  model with central charge  $c = 3(1 - 2/p)$ .

## 1. Introduction

In the past several years it has been observed [1]–[3] that string theories of two dimensional quantum gravity have a strong connection with  $N = 2$  superconformal models. This observation was first made in [1] and is forcefully presented in [3] where it is stated “The underlying  $N = 2$  superconformal symmetry is quite generic and is present in every string theory.”

It has also been realized since the work of Andrews, Baxter, and Forrester [4] and Date, Jimbo, Miwa and Okado [5] that there is a thoroughgoing relationship between Rogers–Ramanujan identities and conformal field theory characters. Consequently the relation seen between  $N = 2$  superconformal field theories and string theory should be expected to be present in the theory of Rogers–Ramanujan identities. It is the purpose of this paper to demonstrate that this is correct and that the characters of  $N = 2$  superconformal models may be obtained from the nonsupersymmetric models  $M(p, p')$  by means of the construction known as the Bailey pair. Indeed it is our contention that this relation between string theory and Bailey pairs is much more than an analogue and that the Bailey construction provides an exact reformulation of the string theory in terms of a completely fermionic description of the Fock spaces.

The method of the Bailey pair was invented by Bailey [6] in his proofs of many of the results of Rogers [7]–[8]. This method was extensively used by Slater [9]–[10], has been extended by Andrews [11], was extended to the Bailey lattice by Agarwal, Andrews and Bressoud [12], and generalized to higher rank groups by Milne and Lilly [13]. Most recently this technique has been combined by Foda and Quano [14] with the finite size results on  $M(p, p + 1)$  of [15]–[16] and  $M(2, 2k + 1)$  of [17] to provide proofs of various fermionic characters formulas conjectured in [18].

By definition a pair of sequences  $(\alpha_n, \beta_n)$  is said to form a Bailey pair (for the group  $A_1$  relative to the parameter  $a$ ) if

$$\beta_n = \sum_{j=0}^n \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}} \quad (1.1)$$

where we use the definition

$$(a; q)_n = \prod_{l=0}^{n-1} (1 - aq^l) \quad (1.2)$$

which, when there is no danger of confusion, we abbreviate as  $(a)_n$ . The inverse of this relation is

$$\alpha_n = \frac{1 - aq^{2n}}{1 - a} \sum_{j=0}^n \frac{(a)_{n+j}(-1)^{n-j} q^{\frac{1}{2}(n-j)(n-j-1)}}{(q)_{n-j}} \beta_j. \quad (1.3)$$

The lemma of Bailey states that whenever  $(\alpha_n, \beta_n)$  is a Bailey pair related by (1.1) then

$$\begin{aligned} & \sum_{n=0}^N \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)_{N-n}(aq/\rho_1\rho_2)^n\beta_n}{(q)_{N-n}(aq/\rho_1)_N(aq/\rho_2)_N} \\ &= \sum_{n=0}^N \left( \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n} \right) \frac{1}{(q)_{N-n}(aq)_{N+n}}. \end{aligned} \quad (1.4)$$

The proofs of these results are given in [6] and [11].

The lemma of Bailey is a result for rational functions. By taking the limit  $N \rightarrow \infty$  we obtain from (1.4) the theorem that if  $(\alpha_n, \beta_n)$  is a Bailey pair then

$$\sum_{n=0}^{\infty} (\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n = \frac{(aq/\rho_1)_{\infty}(aq/\rho_2)_{\infty}}{(aq)_{\infty}(aq/\rho_1\rho_2)_{\infty}} \sum_{n=0}^{\infty} \left( \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n} \right). \quad (1.5)$$

This result has been extensively used by Slater [9]–[10] to prove many of the identities first derived by Rogers [7]–[8] by making the following two specializations of the parameters  $\rho_1, \rho_2$ ;

$$1: \quad \rho_1 \rightarrow \infty, \quad \rho_2 \rightarrow \infty \quad (1.6)$$

$$2: \quad \rho_1 \rightarrow \infty, \quad \rho_2 = \text{finite}, \quad (1.7)$$

and from her extensive list of results [10] one finds the remarkable result that the first specialization (1.6) leads to characters of minimal models  $M(p, p')$  and the second specialization (1.7) leads to characters of  $N = 1$  supersymmetric models  $SM(p, p')$ .

There is, however, a third case not considered by Slater

$$3 : \quad \rho_1 = \text{finite}, \quad \rho_2 = \text{finite}. \quad (1.8)$$

We show here that this leads to characters of the  $N = 2$  supersymmetric models.

It cannot, of course, be an accident that Bailey's construction, invented decades before conformal field theory was even thought about, gives characters of the  $N = 0, 1$  and  $2$  supersymmetric conformal field theories in a unified manner. In fact there is a complete connection between the Bailey construction, the theory of affine Lie algebras and even two dimensional quantum gravity and this paper is meant to be the first of several in which we present these topics in detail. However, there are several technical complications which must be presented in some detail in order to deal with nonunitary models which tend to obscure the emergence of the  $N = 2$  supersymmetry. Consequently in this first paper we will introduce the subject by restricting our attention to the unitary models  $M(p, p+1)$  and their  $N = 1$  and  $N = 2$  supersymmetric extensions.

We begin in sec. 2 by reviewing the construction of Slater [9]–[10] of the characters of  $M(3, 4)$  and  $SM(3, 5)$  and then demonstrate how the characters of the  $N = 2$  unitary model with  $c = 1$  are obtained by using the specialization (1.8). In sec. 3 we present a second construction of Slater [9] which gives  $M(3, 4)$  under the specialization (1.6) but gives a special case of the  $N = 2$  model under (1.7). We finally consider the general unitary model  $M(p, p+1)$  in sec. 4 where we first obtain the Bailey pairs from the finite size polynomial computations of [4], [15], [16], [19]–[21] by use of the method of [14]. We obtain all the characters of  $M(p, p+1)$  from this construction and thus extend the results of [14] and [20]. The relation of this construction, which we call Bailey renormalization flow, to the renormalization flows of [22] (and [23]) is first proposed in [24]. We then obtain from the Bailey pairs the  $N = 1$  models  $SM(p, p+2)$  and the unitary  $N = 2$  models with central charge  $c = 3(1 - 2/p)$ . We conclude in sec. 5 with a few remarks on the more general case which will be presented more fully elsewhere [25].

## 2. The Ising model $M(3, 4)$ and its $N = 1$ and $N = 2$ extensions.

The Ising model is the minimal model  $M(3, 4)$ . It has three independent characters which are obtained from the general bosonic formula for the characters of the  $M(p, p')$  model

$$\chi_{r,s}^{(p,p')}(q) = \chi_{p-r,p'-s}^{(p,p')}(q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (q^{j(pp'+rp'-sp)} - q^{(jp'+s)(jp+r)}). \quad (2.1)$$

Here  $p$  and  $p'$  are relatively prime, the central charge is

$$c = 1 - \frac{6(p - p')^2}{pp'} \quad (2.2)$$

and the conformal dimensions are

$$\Delta_{r,s}^{(p,p')} = \frac{(rp' - sp)^2 - (p - p')^2}{4pp'} \quad (1 \leq r \leq p - 1, 1 \leq s \leq p' - 1). \quad (2.3)$$

These characters also have a fermionic representation as  $q$  series due to the following identities proven in 1894 by Rogers [7]

$$\sum_{m=0, \text{even}}^{\infty} \frac{q^{m^2/2}}{(q)_m} = \chi_{1,1}^{(3,4)}(q), \quad \sum_{m=1, \text{odd}}^{\infty} \frac{q^{m^2/2}}{(q)_m} = \chi_{1,3}^{(3,4)}(q), \quad (2.4)$$

$$\sum_{m=0, \text{even}}^{\infty} \frac{q^{m(m-1)/2}}{(q)_m} = \sum_{m=1, \text{odd}}^{\infty} \frac{q^{m(m-1)/2}}{(q)_m} = \chi_{1,2}^{(3,4)}(q). \quad (2.5)$$

Slater [9] obtains the four identities (2.4) (2.5) from the following four Bailey pairs (which following the notation of Rogers she calls A(5)-A(8)) where  $\alpha_0 = 1$  in all cases and

	$\beta_n$	$\alpha_{3n-1}$	$\alpha_{3n}$	$\alpha_{3n+1}$	$a$
A(5)	$q^{n^2}/(q)_{2n}$	$-q^{3n^2-n}$	$q^{3n^2-n} + q^{3n^2+n}$	$-q^{3n^2+n}$	1
A(6)	$q^{n^2}/(q^2)_{2n}$	$q^{3n^2+n}$	$q^{3n^2-n}$	$-q^{3n^2+n} - q^{3n^2+5n+2}$	$q$
A(7)	$q^{n^2-n}/(q)_{2n}$	$-q^{3n^2-4n+1}$	$q^{3n^2-2n} + q^{3n^2+2n}$	$-q^{3n^2+4n+1}$	1
A(8)	$q^{n^2+n}/(q^2)_{2n}$	$q^{3n^2-2n}$	$q^{3n^2+2n}$	$-q^{3n^2+4n+1} - q^{3n^2+2n}$	$q$

When we put these four pairs into (1.5) we find from A(5) and A(8) that for  $a = 1, q$  (after a bit of algebra)

$$\begin{aligned} \sum_{n=0}^{\infty} (\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^n \frac{q^{n^2} a^n}{(aq)_{2n}} &= \frac{(aq/\rho_1)_{\infty} (aq/\rho_2)_{\infty}}{(aq)_{\infty} (aq/\rho_1 \rho_2)_{\infty}} \sum_{j=-\infty}^{\infty} \\ &\times \left( \frac{(\rho_1)_{3j} (\rho_2)_{3j} (aq/\rho_1 \rho_2)^{3j}}{(aq/\rho_1)_{3j} (aq/\rho_2)_{3j}} - \frac{(\rho_1)_{3j+1} (\rho_2)_{3j+1} (aq/\rho_1 \rho_2)^{3j+1}}{(aq/\rho_1)_{3j+1} (aq/\rho_2)_{3j+1}} \right) a^j q^{3j^2+j} \end{aligned} \quad (2.6)$$

where we define

$$(a)_{-n} = \frac{1}{(1 - aq^{-1})(1 - aq^{-2}) \dots (1 - aq^{-n})} = \frac{1}{(aq^{-n})_n} = \frac{(-q/a)^n q^{\frac{1}{2}n(n-1)}}{(q/a)_n}. \quad (2.7)$$

Similarly we find from A(6) and A(7) that with  $a = 1, q$

$$\sum_{n=-\infty}^{\infty} (\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^n \frac{q^{n(n-1)} a^n}{(aq)_{2n}} = \frac{(aq/\rho_1)_{\infty} (aq/\rho_2)_{\infty}}{(aq)_{\infty} (aq/\rho_1 \rho_2)_{\infty}} \sum_{j=-\infty}^{\infty} a^j q^{3j^2-2j} \quad (2.8)$$

$$\times \left( \frac{(\rho_1)_{3j} (\rho_2)_{3j} (aq/\rho_1 \rho_2)^{3j}}{(aq/\rho_1)_{3j} (aq/\rho_2)_{3j}} - \frac{(\rho_1)_{3j-2} (\rho_2)_{3j-2} (aq/\rho_1 \rho_2)^{3j-2}}{(aq/\rho_1)_{3j-2} (aq/\rho_2)_{3j-2}} \right)$$

These two identities hold for all values of  $\rho_1$  and  $\rho_2$ . By making suitable specializations of these parameters these formulas specialize to the characters of several different conformal field theory models. There are three distinct cases and we will treat them separately.

### 2.1. The model $M(3, 4)$

Slater [10] obtains the Ising model identities (2.4) (2.5) from (2.6) and (2.8) by making the specialization (1.6) where, using

$$\lim_{\rho \rightarrow \infty} \rho^{-n} (\rho)_n = (-1)^n q^{\frac{n}{2}(n-1)} \quad (2.9)$$

we find from (2.6)

$$\sum_{n=0}^{\infty} \frac{q^{2n^2} a^{2n}}{(aq)_{2n}} = \frac{1}{(aq)_{\infty}} \sum_{j=-\infty}^{\infty} \left( q^{j(12j+1)} a^{4j} - q^{(3j+1)(4j+1)} a^{4j+1} \right). \quad (2.10)$$

Then, by setting  $a = 1$  and  $a = q$  we obtain the identities of (2.4) for  $\chi_{1,1}^{(3,4)}(q)$  and  $\chi_{1,3}^{(3,4)}(q)$ . These are (83) and (86) of [10]. Similarly from (2.8) we find

$$\sum_{n=0}^{\infty} \frac{q^{2n^2-n} a^{2n}}{(aq)_{2n}} = \frac{1}{(aq)_{\infty}} \sum_{j=-\infty}^{\infty} \left( q^{12j^2-2j} a^{4j} - q^{(3j+1)(4j+2)} a^{4j+2} \right). \quad (2.11)$$

and by setting  $a = 1$  and  $a = q$  we find the two forms of the identity (2.5) for  $\chi_{1,2}^{(3,4)}(q)$ . These are (84) and (85) of [10].

### 2.2. The $N = 1$ supersymmetric model $SM(3, 5)$

The second set of specializations made by Slater [10] is

$$\rho_1 \rightarrow \infty, \quad \rho_2 = -q^{\frac{1}{2}} \text{ and } -q. \quad (2.12)$$

We will see that these give characters of the  $N = 1$  superconformal model  $SM(p, p')$  where

$$\hat{\chi}_{r,s}^{(p,p')}(q) = \hat{\chi}_{p-r,p'-s}^{(p,p')}(q) = \frac{(-q^{\epsilon_{r-s}})_{\infty}}{(q)_{\infty}} \sum_{j=-\infty}^{\infty} \left( q^{\frac{j(jpp'+rp'-sp)}{2}} - q^{\frac{(jp-r)(jp'-s)}{2}} \right) \quad (2.13)$$

where  $1 \leq r \leq p-1$ ,  $1 \leq s \leq p'-1$ ,  $p$  and  $(p'-p)/2$  are relatively prime and

$$\epsilon_a = \begin{cases} \frac{1}{2} & \text{if } a \text{ is even (Neveu-Schwarz (NS) sector)} \\ 1 & \text{if } a \text{ is odd (Ramond (R) sector).} \end{cases} \quad (2.14)$$

Here the central charge is

$$c = \frac{3}{2} - \frac{3(p-p')^2}{pp'} \quad (2.15)$$

and the conformal dimensions are

$$\frac{(rp' - sp)^2 - (p - p')^2}{8pp'} + \frac{2\epsilon_{r-s}}{16} \quad (2.16)$$

In the first case where  $\rho_2 = -q^{\frac{1}{2}}$  we consider (2.6) with  $a = 1$  and find

$$\sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}})_n q^{\frac{3}{2}n^2}}{(q)_{2n}} = \frac{(-q^{\frac{1}{2}})_{\infty}}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} \left( q^{\frac{1}{2}n(15n-2)} - q^{\frac{1}{2}(3n-1)(5n-1)} \right). \quad (2.17)$$

which from (2.13) is the character  $\hat{\chi}_{1,1}^{3,5}(q)$  of the model  $SM(3,5)$  and is eqn.(100) in Slater's list [10]. Similarly from (2.8) with  $a = 1$

$$\sum_{n=0}^{\infty} (-q^{\frac{1}{2}})_n \frac{q^{\frac{3n^2}{2}-n}}{(q)_{2n}} = \frac{(-q^{\frac{1}{2}})_{\infty}}{(q)_{\infty}} \sum_{j=-\infty}^{\infty} \left( q^{\frac{1}{2}j(15j-4)} - q^{\frac{1}{2}(3j+1)(5j+3)} \right) = \hat{\chi}_{1,3}^{3,5}(q) \quad (2.18)$$

which is (95) on Slater's list.

For the case where  $\rho_2 = -q$  we find from (2.6) with  $a = q$  that

$$\sum_{n=0}^{\infty} (-q)_n \frac{q^{\frac{3n}{2}(n+1)}}{(q)_{2n+1}} = \frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} \left( q^{\frac{1}{2}n(15n+7)} - q^{\frac{1}{2}(3n+2)(5n+1)} \right) = \hat{\chi}_{2,1}^{(3,5)}(q) = \hat{\chi}_{1,4}^{(3,5)}(q) \quad (2.19)$$

which is (63) in Slaters list while setting  $a = q$  in (2.8) we obtain

$$\sum_{n=0}^{\infty} (-q)_n \frac{q^{\frac{n}{2}(3n+1)}}{(q)_{2n+1}} = \frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} \left( q^{\frac{1}{2}n(15n+1)} - q^{\frac{1}{2}(3n+2)(5n+3)} \right) = \hat{\chi}_{2,3}^{(3,5)}(q) = \hat{\chi}_{1,2}^{(3,5)}(q) \quad (2.20)$$

which is (62) of Slater.

These four identities may be put into the canonical quasi particle form [18] by using the elementary identity

$$(x)_n = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q (-x)^j q^{\frac{1}{2}j(j-1)} = \sum_{m=0}^n (-x)^{(n-m)} q^{\frac{1}{2}(n-m)(n-m-1)} \begin{bmatrix} n \\ m \end{bmatrix}_q \quad (2.21)$$

where the  $q$ -binomial coefficient is defined as

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_j (q)_{n-j}} & \text{for } 0 \leq j \leq n \\ 0 & \text{otherwise.} \end{cases} \quad (2.22)$$

Thus we find

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2 + \frac{1}{2}m^2 - nm}}{(q)_{2n}} \begin{bmatrix} n \\ m \end{bmatrix}_q = \hat{\chi}_{1,1}^{(3,5)}(q), \quad (2.23)$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2 + \frac{1}{2}m^2 - nm - n}}{(q)_{2n}} \begin{bmatrix} n \\ m \end{bmatrix}_q = \hat{\chi}_{1,3}^{(3,5)}(q). \quad (2.24)$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2 + \frac{1}{2}m^2 - nm + 2n - \frac{1}{2}m}}{(q)_{2n+1}} \begin{bmatrix} n \\ m \end{bmatrix}_q = \hat{\chi}_{1,4}^{(3,5)}(q), \quad (2.25)$$

and

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2 + \frac{1}{2}m^2 - nm + n - \frac{1}{2}m}}{(q)_{2n+1}} \begin{bmatrix} n \\ m \end{bmatrix}_q = \hat{\chi}_{1,2}^{(3,5)}(q) \quad (2.26)$$

We have now succeeded in obtaining expressions for all four of the independent characters of  $SM(3,5)$ . However just as in the case of  $M(3,4)$  the symmetry condition  $\hat{\chi}_{r,s}^{(p,p')}(q) = \hat{\chi}_{p-r,p'-s}(q)$  implies certain further nontrivial identities. One of these is found by setting  $a = 1$  and  $\rho_2 = -q$  in (2.6) to find a second representation for  $\hat{\chi}_{1,2}^{(3,5)}(q)$  in addition to (2.20) of

$$\sum_{n=0}^{\infty} (-q)_n \frac{q^{\frac{3n^2}{2} - \frac{n}{2}}}{(q)_{2n}} = \hat{\chi}_{1,2}^{(3,5)}(q) \quad (2.27)$$

and if we set  $a = 1$ , and  $\rho_2 = -q$  in (2.8) we obtain

$$\sum_{n=0}^{\infty} (-q)_n \frac{q^{\frac{3n}{2}(n-1)}}{(q)_{2n}} = \hat{\chi}_{1,2}^{(3,5)}(q) + \hat{\chi}_{1,4}^{(3,5)}(q). \quad (2.28)$$

### 2.3. The $N = 2$ unitary supersymmetric model with $c = 1$

We now consider the third specialization (1.8). This was not considered by Slater. We will show that this will lead to a unitary  $N = 2$  supersymmetric model. For these models the central charge is  $c = 3(1 - 2/m)$  and there are three sectors called A for antiperiodic (Neveu-Schwarz), P for periodic (Ramond) and T for twisted. In the A and P sectors the (normalized) characters are [30]-[33]

$$\begin{aligned} & \chi_{r,s}^{(N=2)(m)}(y, q) \\ &= \frac{(-q^{\tilde{\epsilon}}y)_{\infty} (-q^{\tilde{\epsilon}}y^{-1})_{\infty}}{(q)_{\infty}^2} \sum_{j=-\infty}^{\infty} q^{mj^2 + (r+s)j} \left( 1 - \frac{q^{mj+r}y^{-1}}{1 + q^{mj+r}y^{-1}} - \frac{q^{mj+s}y}{1 + q^{mj+s}y} \right) \end{aligned} \quad (2.29)$$



where:

1) In the sector A  $r$  and  $s$  are half integers with  $0 \leq r, s, r+s \leq m-1$  and  $\tilde{\epsilon} = \frac{1}{2}$  and the conformal dimensions of the Virasoro operator  $L_0$  and the  $U(1)$  current  $J_0$  are

$$h_{r,s}^A = (rs - \frac{1}{4})/m, \quad q_{r,s}^A = (r-s)/m; \quad (2.30)$$

2) In the  $P$  sector  $r$  and  $s$  are integers with  $0 \leq r-1, s, r+s \leq m-1$  and  $\tilde{\epsilon} = 1$  and

$$h_{r,s}^P = \frac{rs}{m} + \frac{c}{24}, \quad q_{r,s}^P = (r-s)/m. \quad (2.31)$$

In the T sector the characters are

$$\chi_r^{(N=2,T)(m)}(q) = \frac{(-q^{\frac{1}{2}})_\infty (-q)_\infty}{(q^{\frac{1}{2}})_\infty (q)_\infty} \sum_{j=-\infty}^{\infty} \left( q^{mj^2 + \frac{j}{2}(m-2r)} - q^{(jm-r)(j-\frac{1}{2})} \right) \quad (2.32)$$

where  $1 \leq r \leq m/2$  and

$$h_r^T = \frac{(m-2r)^2}{16m} + \frac{c}{24}. \quad (2.33)$$

In the  $A$  and  $P$  sectors these characters (with  $m=3$ ) are all obtained from the Bailey pairs A(5) and A(8) of Slater by specializing (2.6) to the point

$$aq/\rho_1\rho_2 = 1. \quad (2.34)$$

To carry out this specialization rewrite (2.6) using

$$\begin{aligned} & \frac{(\rho_1)_{3j}(\rho_2)_{3j}(aq/\rho_1\rho_2)^{3j}}{(aq/\rho_1)_{3j}(aq/\rho_2)_{3j}} - \frac{(\rho_1)_{3j+1}(\rho_2)_{3j+1}(aq/\rho_1\rho_2)^{3j+1}}{(aq/\rho_1)_{3j+1}(aq/\rho_2)_{3j+1}} \\ &= \frac{(\rho_1)_{3j}(\rho_2)_{3j}(aq/\rho_1\rho_2)^{3j}}{(aq/\rho_1)_{3j}(aq/\rho_2)_{3j}} \left( 1 - \frac{(1-\rho_1q^{3j})(1-\rho_2q^{3j})(aq/\rho_1\rho_2)}{(1-aq^{3j+1}/\rho_1)(1-aq^{3j+1}/\rho_2)} \right) \\ &= \frac{(\rho_1)_{3j}(\rho_2)_{3j}(aq/\rho_1\rho_2)^{3j}}{(aq/\rho_1)_{3j}(aq/\rho_2)_{3j}} \frac{(1-aq/\rho_1\rho_2)(1-aq^{6j+1})}{(1-aq^{3j+1}/\rho_1)(1-aq^{3j+1}/\rho_2)} \end{aligned} \quad (2.35)$$

and set

$$\rho_1 = -y^{-1}q^r, \quad \rho_2 = -yq^s, \quad a = q^{r+s-1} \quad (2.36)$$

to obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} (-y^{-1}q^r)_n (-yq^s)_n \frac{q^{n^2}q^{n(r+s-1)}}{(q^{r+s})_{2n}} \\ &= \frac{(-y^{-1}q^r)_\infty (-yq^s)_\infty}{(q^{r+s})_\infty (q)_\infty} \sum_{j=-\infty}^{\infty} q^{3j^2+(r+s)j} \left( 1 - \frac{y^{-1}q^{3j+r}}{1+y^{-1}q^{3j+r}} - \frac{yq^{3j+s}}{1+yq^{3j+s}} \right). \end{aligned} \quad (2.37)$$

The right hand side of this agrees with the character formula (2.29) after multiplying by a suitable factor. Thus in the  $A$  sector we obtain the  $q$  series representations

$$\begin{aligned} \sum_{n=0}^{\infty} (-y^{-1}q^{\frac{1}{2}})_n (-yq^{\frac{1}{2}})_n \frac{q^{n^2}}{(q)_{2n}} &= \chi_{\frac{1}{2}, \frac{1}{2}}^{(N=2)(3)}(y, q) \\ \sum_{n=0}^{\infty} (-y^{-1}q^{\frac{1}{2}})_n (-yq^{\frac{1}{2}})_{n+1} \frac{q^{n^2+n}}{(q)_{2n+1}} &= \chi_{\frac{1}{2}, \frac{3}{2}}^{(N=2)(3)}(y, q), \end{aligned} \quad (2.38)$$

and in the  $P$  sector we find the three characters

$$\begin{aligned} \sum_{n=0}^{\infty} (-y^{-1}q)_n (-yq)_n \frac{q^{n^2+n}}{(q)_{2n+1}} &= \chi_{1,1}^{(N=2)(3)}(y, q) \\ \sum_{n=0}^{\infty} (-y^{-1}q)_n (-yq)_{n-1} \frac{q^{n^2}}{(q)_{2n}} &= \chi_{1,0}^{(N=2)(3)}(y, q) \\ \sum_{n=0}^{\infty} (-y^{-1}q)_{n+1} (-yq)_{n-1} \frac{q^{n^2+n}}{(q)_{2n+1}} &= \chi_{2,0}^{(N=2)(3)}(y, q) \end{aligned} \quad (2.39)$$

For the  $T$  sector we specialize (2.6) using

$$\rho_1 = -q^{\frac{1}{2}}, \quad \rho_2 = -q, \quad a = 1 \quad (2.40)$$

to find

$$\sum_{n=0}^{\infty} (-q^{\frac{1}{2}})_n (-q)_n \frac{q^{n^2 - \frac{n}{2}}}{(q)_{2n}} = \chi_1^{(N=2,T)(3)}(q). \quad (2.41)$$

In a like fashion  $q$  series for linear combinations of characters are obtained from the Bailey pairs A(6) and A(7) by specializing (2.8). Thus for  $\rho_1 = -yq^{\frac{1}{2}}$ ,  $\rho_2 = -y^{-1}q^{\frac{1}{2}}$  and  $a = 1$  we find in the  $A$  sector

$$\sum_{n=0}^{\infty} (-y^{-1}q^{\frac{1}{2}})_n (-yq^{\frac{1}{2}})_n \frac{q^{n(n-1)}}{(q)_{2n}} = \chi_{\frac{1}{2}, \frac{3}{2}}^{(N=2)(3)}(y, q) + \chi_{\frac{1}{2}, \frac{3}{2}}^{(N=2)(3)}(y^{-1}, q), \quad (2.42)$$

and for  $\rho_1 = -y^{-1}q$ ,  $\rho_2 = -yq$  and  $a = q$  we find in the  $P$  sector

$$\sum_{n=0}^{\infty} (-y^{-1}q)_n (-yq)_n \frac{q^{n^2}}{(q)_{2n+1}} = \frac{1}{2} \{ \chi_{1,0}^{(N=2)(3)}(y, q) + \chi_{1,0}^{(N=2)(3)}(y^{-1}, q) \}. \quad (2.43)$$

All of these fermionic  $q$ -series may be put in the canonical quasi-particle form [18] by use of (2.21). Thus, for example we find from (2.38) that

$$\begin{aligned} \sum_{m_1, m_2, n=0}^{\infty} y^{m_1 - m_2} q^{\frac{1}{2}(n-m_1)(n-m_1) + \frac{1}{2}(n-m_2)(n-m_2) + n^2} \frac{1}{(q)_{2n}} \begin{bmatrix} n \\ m_1 \end{bmatrix}_q \begin{bmatrix} n \\ m_2 \end{bmatrix}_q \\ = \chi_{\frac{1}{2}, \frac{1}{2}}^{(N=2)(3)}(y, q) \end{aligned} \quad (2.44)$$

which if we use  $n = m_3/2$  is rewritten as

$$\sum_{\vec{m}=0, m_3 \text{ even}}^{\infty} q^{\frac{1}{4}\vec{m}C_{D_3}\vec{m}} y^{m_1-m_2} \frac{1}{(q)_{m_3}} \left[ \begin{matrix} \frac{1}{2}m_3 \\ m_1 \end{matrix} \right]_q \left[ \begin{matrix} \frac{1}{2}m_3 \\ m_2 \end{matrix} \right]_q = \chi_{\frac{1}{2}, \frac{1}{2}}^{(N=2)(3)}(y, q) \quad (2.45)$$

where  $C_{D_3}$  is the Cartan matrix of  $D_3$ . This form of the  $N = 2$  characters was first given in [18].

### 3. Supersymmetry from the $F$ pairs of Slater

Slater [9], in fact, provides not one but two constructions of the characters of the Ising model. Her second construction is from the pairs she calls  $F$  and is quite different from the one presented above, not only in that the bosonic form of the  $M(3, 4)$  character is not of the Rocha-Caridi form (2.1) but the specialization (1.7) which for the  $A$  pairs gave  $N = 1$  supersymmetry now gives  $N = 2$ .

#### 3.1. The Bailey pairs

We will derive a set of Bailey pairs which are somewhat more general than Slater [9] in that our form will be valid for all values of  $a$  whereas hers are valid only for  $a = 0$  and 1. In particular we start with her equation (2.1) and set  $e = a$ ,  $b = q^{-j}$ , and use the identity

$$\frac{(x)_{\infty}}{(xq^j)_{\infty}} = (x)_j \quad (3.1)$$

to find

$$\sum_{n=0}^j \frac{(1 - aq^{2n})(q^{-j})_n (c)_n (d)_n (a)_n}{(1 - a)(aq^{1+j})_n (aq/c)_n (aq/d)_n (q)_n} \left( \frac{aq^{1+j}}{cd} \right)^n = \frac{(aq)_j (aq/cd)_j}{(aq/c)_j (aq/d)_j} \quad (3.2)$$

and then use

$$(aq)_{N+n} = (aq)_N (aq^{N+1})_n \quad (3.3)$$

and

$$(xq^{-N})_n = (-1)^n x^n q^{-nN + \frac{1}{2}n(n-1)} \frac{(q/x)_N}{(q/x)_{N-n}} \quad (3.4)$$

to obtain

$$\sum_{n=0}^j \frac{(1 - aq^{2n})(-1)^n q^{\frac{1}{2}n(n+1)} a^n (c)_n (d)_n (a)_n}{(1 - a)(aq)_{n+j} (q)_{j-n} (q)_n c^n (aq/c)_n d^n (aq/d)_n} = \frac{(aq/cd)_j}{(q)_j (aq/c)_j (aq/d)_j}. \quad (3.5)$$

There are many specializations of this result which will give Bailey pairs. For example if  $c, d \rightarrow \infty$  we obtain Bailey pairs for the original Rogers-Ramanujan identities for the  $M(2, 5)$  model.

We here consider first  $c = q^{\frac{1}{2}}$  and  $d \rightarrow \infty$  to find

$$\sum_{n=0}^j \frac{(1 - aq^{2n})q^{n^2 - \frac{n}{2}} a^n (a)_n (q^{\frac{1}{2}})_n}{(1 - a)(aq)_{n+j} (q)_{j-n} (q)_n (aq^{\frac{1}{2}})_n} = \frac{1}{(q)_j (aq^{\frac{1}{2}})_j} \quad (3.6)$$

and thus comparing with the definition of Bailey pair (1.1) we find the Bailey pair

$$\begin{aligned} \alpha_n &= \frac{(1 - aq^{2n})q^{n^2 - \frac{n}{2}} a^n (a)_n (q^{\frac{1}{2}})_n}{(1 - a)(q)_n (aq^{\frac{1}{2}})_n} \\ \beta_n &= \frac{1}{(q)_n (aq^{\frac{1}{2}})_n}. \end{aligned} \quad (3.7)$$

We note in particular that Slater gives the special case  $F(1)$  of  $a = 1$  where

$$\beta_n = \frac{1}{(q^{\frac{1}{2}})_n (q)_n}, \quad \alpha_n = \begin{cases} 1 & \text{for } n = 0 \\ q^{n^2} (q^{\frac{n}{2}} + q^{-\frac{n}{2}}) & \text{for } n \geq 1 \end{cases} \quad (3.8)$$

and  $F(2)$  of  $a = q$  where

$$\beta_n = \frac{1}{(q^{\frac{3}{2}})_n (q)_n}, \quad \alpha_n = q^{n^2 + \frac{n}{2}} \frac{1 + q^{n + \frac{1}{2}}}{1 + q^{\frac{1}{2}}}. \quad (3.9)$$

We now use the Bailey (3.7) pair in Bailey's lemma (1.5) and obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^n}{(q)_n (aq^{\frac{1}{2}})_n} \\ &= \frac{(aq/\rho_1)_{\infty} (aq/\rho_2)_{\infty}}{(aq)_{\infty} (aq/\rho_1 \rho_2)_{\infty}} \sum_{n=0}^{\infty} \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^n}{(aq/\rho_1)_n (aq/\rho_2)_n} \frac{(1 - aq^{2n})q^{n^2 - \frac{n}{2}} a^n (a)_n (q^{\frac{1}{2}})_n}{(1 - a)(q)_n (aq^{\frac{1}{2}})_n}. \end{aligned} \quad (3.10)$$

As in sec. 2 we will obtain characters of conformal field theory by considering different specializations of the parameters  $\rho_1$ , and  $\rho_2$ .

### 3.2. The case $\rho_1$ and $\rho_2 \rightarrow \infty$ ; the model $M(3, 4)$

The first specialization to consider is  $\rho_1, \rho_2 \rightarrow \infty$  where we find from (3.10) that

$$\sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q)_n (aq^{\frac{1}{2}})_n} = \frac{1}{(aq)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{2n^2 - \frac{n}{2}} a^{2n} (1 - aq^{2n}) (a)_n (q^{\frac{1}{2}})_n}{(1 - a)(q)_n (aq^{\frac{1}{2}})_n}. \quad (3.11)$$

This result is particularly transparent when  $a = 1$  where we find

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n (q^{\frac{1}{2}})_n} = \frac{1}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} q^{2n^2} q^{\frac{n}{2}} \quad (3.12)$$

from which if we finally send  $q \rightarrow q^2$  and use the identity

$$(q^2; q^2)_n (q; q^2)_n = (q)_{2n} \quad (3.13)$$

we obtain

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q)_{2n}} = \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} q^{4n^2+n} = \frac{1}{(q)_{\infty} (-q)_{\infty}} \sum_{n=-\infty}^{\infty} q^{4n^2+n} \quad (3.14)$$

which is (39) of Slater (3.1). From the fermionic side of (2.4) we identify this as the character of the  $M(3, 4)$  model  $\chi_{1,1}^{(3,4)}(q)$  and thus we have produced another bosonic form for the character which is different from that of Rocha-Caridi (2.1) in that the series has no minus signs. Similarly if  $a = q$  we find

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n (q^{\frac{3}{2}})_n} = \frac{1}{(q^2)_{\infty}} \sum_{n=0}^{\infty} q^{2n^2+\frac{3}{2}n} \frac{1+q^{n+\frac{1}{2}}}{1+q^{\frac{1}{2}}} \quad (3.15)$$

and if we again set  $q \rightarrow q^2$  and use

$$(q^2; q^2)_n (q^3; q^2)_n = \frac{(q)_{2n+1}}{1-q} \quad (3.16)$$

we find

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q)_{2n+1}} = \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{4n^2+3n} (1+q^{2n+1}) = \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} q^{4n^2+3n} \quad (3.17)$$

where the final expression is obtained from the previous sum by using  $n \rightarrow -n-1$  in the second term. This identity is (38) of Slater [10] and on comparison of the fermionic side with (2.4) is seen to be the character  $\chi_{1,3}^{(3,4)}(q)$ .

We have thus obtained new bosonic sum expressions for the characters  $\chi_{1,1}^{(3,4)}(q)$  and  $\chi_{1,3}^{(3,4)}(q)$  of the model  $M(3, 4)$ . It remains to find the bosonic forms for the character  $\chi_{1,2}^{(3,4)}(q)$ . This is done [9] by returning to (3.5) and setting  $c = q^{\frac{1}{2}}$  and  $d = 0$  to find

$$\sum_{n=0}^j \frac{(1-aq^{2n})q^{-\frac{n}{2}}(q^{\frac{1}{2}})_n(a)_n}{(1-a)(aq)_{n+j}(q)_{j-n}(q)_n(aq^{\frac{1}{2}})_n} = \frac{q^{-\frac{j}{2}}}{(q)_j(aq^{\frac{1}{2}})_j}. \quad (3.18)$$

Thus we find the Bailey pairs

$$\alpha_n = \frac{(1 - aq^{2n})q^{-\frac{n}{2}}(q^{\frac{1}{2}})_n(a)_n}{(1 - a)(q)_n(aq^{\frac{1}{2}})_n}, \quad \beta_n = \frac{q^{-\frac{n}{2}}}{(q)_n(aq^{\frac{1}{2}})_n} \quad (3.19)$$

and in particular note that for  $a = 1$  we have

$$\beta_n = \frac{q^{-\frac{n}{2}}}{(q)_n(q^{\frac{1}{2}})_n}, \quad \alpha_n = \begin{cases} 1 & \text{for } n = 0 \\ q^{\frac{n}{2}} + q^{-\frac{n}{2}} & \text{for } n \geq 1 \end{cases} \quad (3.20)$$

which is F(3) of Slater [9] and setting  $a = q$  we find

$$\beta_n = \frac{q^{-\frac{n}{2}}}{(q)_n(q^{\frac{3}{2}})_n}, \quad \alpha_n = q^{-\frac{n}{2}} \frac{1 + q^{n+\frac{1}{2}}}{1 + q^{\frac{1}{2}}} \quad (3.21)$$

which is F(4) of Slater. Thus if we use this in Bailey's lemma with  $N \rightarrow \infty$  we find

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n(aq^{\frac{1}{2}}/\rho_1\rho_2)^n}{(q)_n(aq^{\frac{1}{2}})_n} \\ &= \frac{(aq/\rho_1)_{\infty}(aq/\rho_2)_{\infty}}{(aq)_{\infty}(aq/\rho_1\rho_2)_{\infty}} \sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n} \frac{(1 - aq^{2n})q^{-\frac{n}{2}}(q^{\frac{1}{2}})_n(a)_n}{(1 - a)(q)_n(aq^{\frac{1}{2}})_n} \end{aligned} \quad (3.22)$$

and then sending  $\rho_1, \rho_2 \rightarrow \infty$  we find

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n/2}a^n}{(q)_n(aq^{\frac{1}{2}})_n} = \frac{1}{(aq)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{n^2-\frac{n}{2}}a^n(1 - aq^{2n})(a)_n(q^{\frac{1}{2}})_n}{(1 - a)(q)_n(aq^{\frac{1}{2}})_n}. \quad (3.23)$$

Then if  $a = 1$  and  $q \rightarrow q^2$  we use (3.10) to find

$$\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q)_{2n}} = \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} q^{2n^2} q^n \quad (3.24)$$

where from (2.4) we identify the fermionic side as  $\chi_{1,2}^{(3,4)}(q)$ . Similarly if  $a = q$  and  $q \rightarrow q^2$  we find

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q)_{2n+1}} = \frac{1}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{2n^2+n}(1 + q^{2n+1}) \quad (3.25)$$

and we note that if in the second term of the bosonic sum we let  $n \rightarrow -n - 1$  we regain the bosonic sum of (3.24). This agrees with the equality of the characters  $\chi_{1,2}^{(3,4)}(q)$  and  $\chi_{2,2}^{(3,4)}(q)$  which is seen in (2.4) and also agrees with (9) of Slater [10].

### 3.3. The case $\rho_1 \rightarrow \infty$ and $\rho_2$ finite ; $N = 2$ supersymmetry

We next let  $\rho_1 \rightarrow \infty$  and  $\rho_2 = -q^{\frac{1}{2}}$ . Then for the Bailey pair (3.7) we obtain from (3.10)

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}} a^n (-q^{\frac{1}{2}})_n}{(q)_n (aq^{\frac{1}{2}})_n} = \frac{(-aq^{\frac{1}{2}})_{\infty}}{(aq)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2}{2} - \frac{n}{2}} a^{2n} (1 - aq^{2n}) (a)_n (q^{\frac{1}{2}})_n (-q^{\frac{1}{2}})_n}{(1-a)(q)_n (aq^{\frac{1}{2}})_n (-aq^{\frac{1}{2}})_n}. \quad (3.26)$$

Then if  $a = 1$  and  $q \rightarrow q^2$  we find

$$\sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(q)_{2n}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} q^{3n^2+n} \quad (3.27)$$

where the fermionic side is recognized as (29) of Slater [10]. Moreover, if we use the identity

$$\begin{aligned} (-q; q^2)_n &= \prod_{l=0}^{n-1} (1 + q^{2l+1}) = \prod_{l=0}^{n-1} (1 + iq^{\frac{1}{2}} q^l) (1 - iq^{\frac{1}{2}} q^l) \\ &= (iq^{\frac{1}{2}})_n (-iq^{\frac{1}{2}})_n \end{aligned} \quad (3.28)$$

we may rewrite the fermionic side to obtain

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(iq^{\frac{1}{2}})_n (-iq^{\frac{1}{2}})_n}{(q)_{2n}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} q^{3n^2+n} \quad (3.29)$$

and thus comparing the fermionic side with (2.38) with  $y = i$  we identify this with the  $N = 2$  supersymmetric character  $\chi_{\frac{1}{2}, \frac{1}{2}}^{(N=2)(m=3)}(i, q)$ . This identification has not appeared in the literature before.

Similarly if  $a = q$  and  $q \rightarrow q^2$  we find the  $q$ -series identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n} (-q; q^2)_n}{(q)_{2n+1}} = \sum_{n=0}^{\infty} q^{n^2+2n} \frac{(iq^{\frac{1}{2}})_n (-iq^{\frac{1}{2}})_n}{(q)_{2n+1}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{3n^2+3n}. \quad (3.30)$$

This last expression, however, is not a theta function and lacks the physical interpretation as a character.

We may also consider the Bailey pair (3.7) with  $\rho_1 \rightarrow \infty$  and  $\rho_2 = -q$  where we find from (3.10)

$$\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} a^n (-q)_n}{(q)_n (aq^{\frac{1}{2}})_n} = \frac{(-a)_{\infty}}{(aq)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{\frac{3n^2}{2} - n} a^{2n} (1 - aq^{2n}) (a)_n (q^{\frac{1}{2}})_n (-q)_n}{(1-a)(q)_n (aq^{\frac{1}{2}})_n (-a)_n}. \quad (3.31)$$

Then setting  $a = 1$  and  $q \rightarrow q^2$  we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-q^2; q^2)_n}{(q)_{2n}} = \sum_{n=0}^{\infty} q^{n(n-1)} \frac{(iq)_n(-iq)_n}{(q)_{2n}} = \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{3n^2} (q^n + q^{-n})^2 \quad (3.32)$$

which is also an identity is not found in Slater and which lacks a character interpretation.

Similarly if  $a = q$  and  $q \rightarrow q^2$  we find

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2; q^2)_n}{(q)_{2n+1}} &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(iq)_n(-iq)_n}{(q)_{2n+1}} = \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{3n^2+2n} (1 + q^{2n+1}) \\ &= \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} q^{3n^2+2n} \end{aligned} \quad (3.33)$$

Here the fermionic side is (28) of Slater and is also the fermionic side of  $\chi_{1,1}^{(N=2)(3)}(i, q)$  (2.39).

Similarly for the Bailey pair (3.19) if we set  $\rho_1 \rightarrow \infty$  and  $\rho_2 = -q^{\frac{1}{2}}$  in (3.22) we find

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n}{2}(n-1)} a^n (-q^{\frac{1}{2}})_n}{(q)_n (aq^{\frac{1}{2}})_n} = \frac{(-aq^{\frac{1}{2}})_{\infty}}{(aq)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2} - \frac{n}{2}} a^n (1 - aq^{2n}) (a)_n (q^{\frac{1}{2}})_n (-q^{\frac{1}{2}})_n}{(1-a)(q)_n (aq^{\frac{1}{2}})_n (-aq^{\frac{1}{2}})_n}. \quad (3.34)$$

Thus, if  $a = 1$  and  $q \rightarrow q^2$  we find

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-q; q^2)_n}{(q)_{2n}} = \sum_{n=0}^{\infty} q^{n(n-1)} \frac{(iq^{\frac{1}{2}})_n(-iq^{\frac{1}{2}})_n}{(q)_{2n}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} q^{n^2+n} \quad (3.35)$$

and by comparison of the fermionic side with (2.42) we see that this is the sum of characters  $\chi_{\frac{1}{2}, \frac{3}{2}}^{(N=2)(3)}(i, q) + \chi_{\frac{3}{2}, \frac{1}{2}}^{(N=2)(3)}(i, q)$ . This result is not found in Slater [10].

Finally we set  $\rho_1 = -q$  and  $\rho_2 \rightarrow \infty$  in (3.22) and find for  $a = q$  and  $q \rightarrow q^2$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^2; q^2)_n}{(q)_{2n+1}} = \sum_{n=0}^{\infty} q^{n^2} \frac{(iq)_n(-iq)_n}{(q)_{2n+1}} = \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} q^{n^2} \quad (3.36)$$

where the fermionic side is identical with that of (2.43) with  $y = i$  and thus is equal to  $\frac{1}{2} \left( \chi_{1,0}^{(N=2)(3)}(i, q) + \chi_{1,0}^{(N+2)(3)}(-i, q) \right)$ . This result also is not found in Slater.



#### 4. $N = 2$ supersymmetric models from $M(p-1, p)$

The derivation given in the previous section of the characters of a  $N = 1$  and  $N = 2$  supersymmetric model from the  $M(3, 4)$  minimal model by means of the Bailey lemma may be generally applied to the  $M(p, p')$  model. We will however restrict ourselves here to the unitary case  $M(p-1, p)$  and treat the general case in [25]. In order to somewhat simplify our treatment we will first extend Bailey's lemma (1.4) to what we will call the bilateral Bailey lemma. We will then demonstrate how from the Bailey pair obtained from the finite size character polynomials for  $M(p-1, p)$  we may construct the characters for  $M(p, p+1)$ . This is an example of the interpretation of renormalization flows [22]–[23] in terms of Bailey pairs mentioned in [24]. We then generalize the method of sec. 2 to obtain the  $N = 1$  and  $N = 2$  supersymmetric models by the specialization of parameters  $\rho_1$  and  $\rho_2$ .

##### 4.1. Bilateral Bailey Pairs

The pair  $(\alpha_n, \beta_n)$  is said to be a bilateral Bailey pair if the following relation is satisfied

$$\beta_n = \sum_{j=-\infty}^n \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}} \quad (4.1)$$

where  $(a)_{-n}$  is defined as in (2.7). One can easily check that with this definition the following important properties of  $(a)_n$  hold for positive and negative integer  $n, k$

$$\begin{aligned} \frac{(a)_n}{(a)_{n-k}} &= (-1)^k (a/q)^k q^{nk - \frac{1}{2}k(k-1)} (q^{1-n}/a)_k, \\ (a)_{n+k} &= (a)_n (aq^n)_k. \end{aligned} \quad (4.2)$$

Analogous to the proof of the original Bailey lemma (1.4) given in [6] and [11] one may derive the bilateral Bailey lemma using (4.2) which states that if  $(\alpha_n, \beta_n)$  satisfy (4.1) then

$$\begin{aligned} \sum_{n=-\infty}^N \left( \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^n \alpha_n}{(aq/\rho_1)_n (aq/\rho_2)_n} \right) \frac{1}{(q)_{N-n} (aq)_{N+n}} \\ = \sum_{n=-\infty}^N \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^{N-n} (aq/\rho_1 \rho_2)^n \beta_n}{(q)_{N-n} (aq/\rho_1)_N (aq/\rho_2)_N} \end{aligned} \quad (4.3)$$

given that the series converge. In analogy to (1.5) we let  $N \rightarrow \infty$  to obtain

$$\sum_{n=-\infty}^{\infty} (\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^n \beta_n = \frac{(aq/\rho_1)_{\infty} (aq/\rho_2)_{\infty}}{(aq)_{\infty} (aq/\rho_1 \rho_2)_{\infty}} \sum_{n=-\infty}^{\infty} \left( \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^n \alpha_n}{(aq/\rho_1)_n (aq/\rho_2)_n} \right). \quad (4.4)$$

Following Andrews [11] one may introduce dual (bilateral) Bailey pairs. If  $(\alpha_n, \beta_n)$  is a (bilateral) Bailey pair relative to  $a$  then

$$\begin{aligned} A_n(a, q) &= a^n q^{n^2} \alpha_n(a^{-1}, q^{-1}), \\ B_n(a, q) &= a^{-n} q^{-n^2-n} \beta_n(a^{-1}, q^{-1}) \end{aligned} \quad (4.5)$$

also satisfy (1.1) ((4.1)) relative to  $a$  and  $(A_n, B_n)$  is called the dual (bilateral) Bailey pair to  $(\alpha_n, \beta_n)$ .

Foda-Quano [14] used the polynomial Fermi-Bose character identity for the  $M(p-1, p)$  model to derive Bailey pairs. We will quickly review their method and slightly generalize it to obtain Bailey pairs that yield the characters for the  $M(p, p+1)$  model, the  $SM(p, p+2)$  model and the  $N=2$  supersymmetric model with central charge  $c=3(1-2/p)$ .

The Bose-Fermi character polynomial identities for the minimal models  $M(p-1, p)$  are of the form

$$B_{r,s}^{(L,p)} = F_{r,s}^{(L,p)} \quad (4.6)$$

where  $B_{r,s}^{(L,p)}$  is the function of Andrews, Baxter and Forrester [4]

$$\begin{aligned} B_{r,s}^{(L,p)} &= \sum_{j=-\infty}^{\infty} \left( q^{j(jp(p-1)+pr-(p-1)s)} \left[ \begin{matrix} L \\ [\frac{1}{2}(L+s-r)] - pj \end{matrix} \right]_q \right. \\ &\quad \left. - q^{(jp-s)(j(p-1)-r)} \left[ \begin{matrix} L \\ [\frac{1}{2}(L-s-r)] + pj \end{matrix} \right]_q \right). \end{aligned} \quad (4.7)$$

Here  $\begin{bmatrix} n \\ j \end{bmatrix}_q$  are the  $q$ -binomial coefficients defined in (2.22) and  $[x]$  denotes the integer part of  $x$ . Equation (4.7) can be put in the form of (4.1) by setting  $L = 2l + r - s + 2x$  in (4.7) and using

$$\begin{aligned} \left[ \begin{matrix} 2l + r - s + 2x \\ l + x - pj \end{matrix} \right]_q &= \frac{(q)_{2l+r-s+2x}}{(q)_{l-(pj-x)}(q)_{l+r-s+x+pj}} \\ &= \frac{(q^{r-s+2x+1})_{2l}}{(q)_{l-(pj-x)}(q^{r-s+2x+1})_{l+(pj-x)}}, \end{aligned} \quad (4.8)$$

$$\left[ \begin{matrix} 2l + r - s + 2x \\ l + x - s + pj \end{matrix} \right]_q = \frac{(q^{r-s+2x+1})_{2l}}{(q)_{l-(pj-r-x)}(q^{r-s+2x+1})_{l+(pj-r-x)}}.$$

Thus we can read off the Bailey pair relative to  $a = q^{r-s+2x}$  from (4.6) as

$$\alpha_n = \begin{cases} q^{j(jp(p-1)+pr-(p-1)s)} & \text{for } n = pj - x \\ -q^{(jp-s)(j(p-1)-r)} & \text{for } n = pj - r - x \\ 0 & \text{otherwise} \end{cases} \quad (4.9)$$

$$\beta_n = \begin{cases} \frac{1}{(aq)_{2n}} F_{r,s}^{(2n+r-s+2x,p)}(q) & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly if we set  $L = 2l+r-s+2x+1$  we get the Bailey pair relative to  $a = q^{r-s+2x+1}$

$$\alpha_n = \begin{cases} q^{j(jp(p-1)+pr-(p-1)s)} & \text{for } n = pj - x \\ -q^{(jp-s)(j(p-1)-r)} & \text{for } n = pj - r - x - 1 \\ 0 & \text{otherwise} \end{cases} \quad (4.10)$$

$$\beta_n = \begin{cases} \frac{1}{(aq)_{2n}} F_{r,s}^{(2n+r-s+2x+1,p)}(q) & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In addition the dual Bailey pairs are obtained from the Bailey dual construction (4.5). Accordingly we get the two dual Bailey pairs, one relative to  $a = q^{r-s+2x}$  (from (4.9))

$$\alpha_n = \begin{cases} q^{j^2 p-sj} q^{x(s-r-x)} & \text{for } n = pj - x \\ -q^{j^2 p-sj} q^{x(s-r-x)} & \text{for } n = pj - r - x \\ 0 & \text{otherwise} \end{cases} \quad (4.11)$$

$$\beta_n = \begin{cases} \frac{1}{(aq)_{2n}} q^{n^2} a^n F_{r,s}^{(2n+r-s+2x,p)}(q^{-1}) & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and the second relative to  $a = q^{r-s+2x+1}$  (from (4.10))

$$\alpha_n = \begin{cases} q^{j^2 p+pj-js} q^{x(s-r-x-1)} & \text{for } n = pj - x \\ -q^{(j-1)(jp-s)} q^{x(s-r-x-1)} & \text{for } n = pj - r - x - 1 \\ 0 & \text{otherwise} \end{cases} \quad (4.12)$$

$$\beta_n = \begin{cases} \frac{1}{(aq)_{2n}} q^{n^2} a^n F_{r,s}^{(2n+r-s+2x+1,p)}(q^{-1}) & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus far we have explicitly constructed only the bosonic side of the Bailey pair. The fermionic side for the  $M(p-1, p)$  model

$$\begin{aligned} F_{r,s}^{(L,p)} &= q^{-(s-r)(s-r-1)/4} \sum_{\vec{m} \equiv \vec{Q}_{r,s}^{(p-3)}} q^{\frac{1}{4}\vec{m}C_{p-3}\vec{m} - \frac{1}{2}\vec{A}_{r,s}^{(p-3)}\vec{m}} \left[ \begin{matrix} \frac{1}{2}(m_2 + L + (\vec{u}_{r,s}^{(p-3)})_1) \\ m_1 \end{matrix} \right]_q \\ &\times \left( \prod_{i=2}^{p-4} \left[ \begin{matrix} \frac{1}{2}(m_{i-1} + m_{i+1} + (\vec{u}_{r,s}^{(p-3)})_i) \\ m_i \end{matrix} \right]_q \right) \left[ \begin{matrix} \frac{1}{2}(m_{p-4} + (\vec{u}_{r,s}^{(p-3)})_{p-3}) \\ m_{p-3} \end{matrix} \right]_q \end{aligned} \quad (4.13)$$

has been proven in [16] , [19]-[21]. Here  $C_{p-3}$  is the  $(p-3) \times (p-3)$  dimensional Cartan matrix of the Lie algebra  $A_{p-3}$  with the elements  $C_{j,k} = 2\delta_{j,k} - \delta_{j-1,k} - \delta_{j+1,k}$ . Furthermore we define the vectors  $\vec{e}_i$  as the  $(p-3)$ -dimensional vectors of unit length in the  $i^{\text{th}}$  direction

$$(\vec{e}_i)_j = \begin{cases} \delta_{i,j} & \text{if } 1 \leq j \leq p-3 \\ 0 & \text{otherwise} \end{cases} \quad (4.14)$$

The vectors  $\vec{A}_{r,s}^{(p-3)}, \vec{u}_{r,s}^{(p-3)} \in \mathbf{N}^{p-3}$  and  $\vec{Q}_{r,s}^{(p-3)} \in (\mathbf{Z}_2)^{p-3}$ . The sum in (4.13) runs over  $\vec{m} \in \mathbf{N}^{p-3}$  such that  $\vec{m} \equiv \vec{Q}_{r,s}^{(p-3)} \pmod{2}$ . When  $L+r-s$  even

$$\begin{aligned} \vec{A}_{r,s}^{(p-3)} &= \vec{e}_{s-1} \\ \vec{u}_{r,s}^{(p-3)} &= \vec{e}_{s-1} + \vec{e}_{p-r-1} \\ \vec{Q}_{r,s}^{(p-3)} &= (r-1)\vec{\rho}^{(p-3)} + (\vec{e}_{s-2} + \vec{e}_{s-4} + \dots) + (\vec{e}_{p-r} + \vec{e}_{p+2-r} + \dots) \end{aligned} \quad (4.15)$$

and when  $L+r-s$  odd

$$\begin{aligned} \vec{A}_{r,s}^{(p-3)} &= \vec{e}_{p-s-1} \\ \vec{u}_{r,s}^{(p-3)} &= \vec{e}_{p-s-1} + \vec{e}_r \\ \vec{Q}_{r,s}^{(p-3)} &= (s-1)\vec{\rho}^{(p-3)} + (\vec{e}_{r-1} + \vec{e}_{r-3} + \dots) + (\vec{e}_{p-s} + \vec{e}_{p+2-s} + \dots) \end{aligned} \quad (4.16)$$

where  $\vec{\rho}^{(p-3)} = \vec{e}_1 + \dots + \vec{e}_{p-3}$ .

Using

$$\lim_{L \rightarrow \infty} \begin{bmatrix} L \\ n \end{bmatrix} = \frac{1}{(q)_n} \quad (4.17)$$

one may take the limit  $L \rightarrow \infty$  in (4.13)

$$F_{r,s}^{(\infty,p)} = q^{-\frac{1}{4}(s-r)(s-r-1)} \sum_{\vec{m} \equiv \vec{Q}_{r,s}^{(p-3)}} q^{\frac{1}{4}\vec{m}C_{p-3}\vec{m} - \frac{1}{2}\vec{A}_{r,s}^{(p-3)}\vec{m}} \frac{1}{(q)_{m_1}} \prod_{i=2}^{p-3} \left[ \frac{\frac{1}{2}(I_{p-3}\vec{m} + \vec{u}_{r,s}^{(p-3)})_i}{m_i} \right] \quad (4.18)$$

where we introduced the incidents matrix  $I_{p-3} = 2 - C_{p-3}$  for compact notation. Notice that there are two ways to take the limit  $L \rightarrow \infty$ , such that  $L+r-s$  even or such that  $L+r-s$  odd. In the limit  $L+r-s$  even  $\vec{A}_{r,s}^{(p-3)}, \vec{u}_{r,s}^{(p-3)}, \vec{Q}_{r,s}^{(p-3)}$  are given as in (4.15) and for  $L+r-s$  odd as in (4.16). Both limits yield the same  $q$ -series and the two different sets for  $\vec{A}_{r,s}^{(p-3)}, \vec{u}_{r,s}^{(p-3)}$  and  $\vec{Q}_{r,s}^{(p-3)}$  reflect the symmetry of the characters  $\chi_{r,s}^{(p-1,p)} = \chi_{p-1-r,p-s}^{(p-1,p)}$ .

We now use the dual bilateral Bailey pairs (4.11) and (4.12) in the bilateral Bailey lemma (4.4). Thus if we use (4.11) in the bilateral Bailey lemma (4.4) we find

$$\begin{aligned} & \frac{(aq/\rho_1)_\infty (aq/\rho_2)_\infty}{(aq)_\infty (aq/\rho_1\rho_2)_\infty} \sum_{j=-\infty}^{\infty} q^{j(jp-s)} q^{x(s-r-x)} \\ & \times \left( \frac{(\rho_1)_{pj-x} (\rho_2)_{pj-x} (aq/\rho_1\rho_2)^{pj-x}}{(aq/\rho_1)_{pj-x} (aq/\rho_2)_{pj-x}} - \frac{(\rho_1)_{pj-r-x} (\rho_2)_{pj-r-x} (aq/\rho_1\rho_2)^{pj-r-x}}{(aq/\rho_1)_{pj-r-x} (aq/\rho_2)_{pj-r-x}} \right) \\ & = \sum_{n=0}^{\infty} (\rho_1)_n (\rho_2)_n (aq/\rho_1\rho_2)^n \frac{q^{n^2} a^n}{(aq)_{2n}} F_{r,s}^{(2n+r-s+2x,p)}(q^{-1}) \end{aligned} \quad (4.19)$$

where  $a = q^{r-s+2x}$ . A very similar equation holds if we insert (4.12) in the bilateral Bailey lemma (4.4).

We now need to consider three specializations of the parameters  $\rho_1, \rho_2$  as done in section 2.

#### 4.2. The model $M(p, p+1)$

The first specialization is

$$\rho_1 \rightarrow \infty, \quad \rho_2 \rightarrow \infty. \quad (4.20)$$

Then if we also set  $x = 0$  we obtain from (4.19) with  $a = q^{r-s}$

$$\begin{aligned} & \frac{1}{(q^{r-s+1})_\infty} \sum_{j=-\infty}^{\infty} \left( q^{j(jp(p+1)+rp-s(p+1))} - q^{(jp-s)(j(p+1)-r)} \right) \\ & = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n(r-s)}}{(q^{r-s+1})_{2n}} F_{r,s}^{(2n+r-s,p)}(q^{-1}) \end{aligned} \quad (4.21)$$

and hence, comparing with (2.1) we find

$$\chi_{r,s}^{(p+1,p)} = \chi_{s,r}^{(p,p+1)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+r-s)}}{(q)_{2n+r-s}} F_{r,s}^{(2n+r-s,p)}(q^{-1}). \quad (4.22)$$

Moreover since we started out with the model  $M(p-1, p)$   $r$  and  $s$  are restricted to  $1 \leq r \leq p-2$ ,  $1 \leq s \leq p-1$  whereas we need (4.22) for  $1 \leq r \leq p$  and  $1 \leq s \leq p-1$ . Thus, the range of  $r$  needed for  $\chi_{s,r}^{(p,p+1)}$  is larger than the range of  $r$  for which (4.22) holds.

To get the remaining characters we make use of the Bailey pair (4.12) with  $x = 0$  and  $a = q^{r-s+1}$ . Then using the bilateral Bailey lemma (4.4) with  $\rho_1, \rho_2 \rightarrow \infty$  we obtain

$$\chi_{s,r+2}^{(p,p+1)} = \sum_{n=0}^{\infty} \frac{q^{2n(n+r-s+1)}}{(q)_{2n+r-s+1}} F_{r,s}^{(2n+r-s+1,p)}(q^{-1}). \quad (4.23)$$

As an application of these results we may check whether (4.22) and (4.23) yield the known expressions for the fermionic characters. To this end let us first compute  $F_{r,s}^{(L,p)}(q^{-1})$ . Observe that

$$\begin{bmatrix} n \\ m \end{bmatrix}_{q^{-1}} = q^{m(m-n)} \begin{bmatrix} n \\ m \end{bmatrix}_q \quad (4.24)$$

from which it follows from (4.13) that

$$F_{r,s}^{(L,p)}(q^{-1}) = q^{(s-r)(s-r-1)/4} \sum_{\vec{m} \equiv \vec{Q}_{r,s}^{(p-3)}} q^{\frac{1}{4}\vec{m}C_{p-3}\vec{m} + \frac{1}{2}\vec{A}_{r,s}^{(p-3)}\vec{m} - \frac{1}{2}\vec{u}_{r,s}^{(p-3)}\vec{m} - \frac{1}{2}Lm_1} \prod_{i=1}^{p-3} \left[ \frac{\frac{1}{2}(I_{p-3}\vec{m} + \vec{u}_{r,s}^{(p-3)} + L\vec{e}_1)_i}{m_i} \right]_q \quad (4.25)$$

where we introduced the incidence matrix  $I_{p-3} = 2 - C_{p-3}$  for compact notation. Hence we obtain from (4.22) by defining  $m_0 = 2n + r - s$

$$\chi_{s,r}^{(p,p+1)} = q^{(s-r)(s-r-1)/4} \sum_{m_0=0, \text{restr.}}^{\infty} \frac{q^{\frac{1}{2}m_0^2 - \frac{(r-s)^2}{2}}}{(q)_{m_0}} \times \sum_{\vec{m} \equiv \vec{Q}_{r,s}^{(p-3)}} q^{\frac{1}{4}\vec{m}C_{p-3}\vec{m} + \frac{1}{2}\vec{A}_{r,s}^{(p-3)}\vec{m} - \frac{1}{2}\vec{u}_{r,s}^{(p-3)}\vec{m} - \frac{1}{2}m_0m_1} \prod_{i=1}^{p-3} \left[ \frac{\frac{1}{2}(I_{p-3}\vec{m} + \vec{u}_{r,s}^{(p-3)} + m_0\vec{e}_1)_i}{m_i} \right]_q \quad (4.26)$$

where the restrictions on  $m_0$  are such that  $m_0$  even if  $r - s$  even and  $m_0$  odd if  $r - s$  odd and  $\vec{A}_{r,s}^{(p-3)}$ ,  $\vec{u}_{r,s}^{(p-3)}$  and  $\vec{Q}_{r,s}^{(p-3)}$  as given in (4.15). Define  $\vec{m} = (m_0, \vec{m})$ ,  $\vec{Q}_{r,s}$ ,  $\vec{u}_{r,s}$ ,  $\vec{A}_{r,s}$  as in (4.15) but where now  $(\vec{e}_i)_j = \delta_{ij}$  for  $0 \leq i \leq p-3$  and otherwise zero and  $C_{p-2}$  as the  $(p-2)$  dimensional Cartan matrix of the Lie algebra  $A_{p-2}$ . Accordingly  $I_{p-2} = 2 - C_{p-2}$ . Then we have  $\frac{1}{4}\vec{m}C_{p-2}\vec{m} = \frac{1}{2}m_0^2 - \frac{1}{2}m_0m_1 + \frac{1}{4}\vec{m}C_{p-3}\vec{m}$ . Hence

$$\chi_{s,r}^{(p,p+1)} = q^{-\frac{1}{4}(r-s)(r-s-1)} \sum_{\vec{m} \equiv \vec{Q}_{r,s}} \frac{1}{(q)_{m_0}} q^{\frac{1}{4}\vec{m}C_{p-2}\vec{m} + \frac{1}{2}\vec{A}_{r,s}\vec{m} - \frac{1}{2}\vec{u}_{r,s}\vec{m}} \prod_{i=1}^{p-3} \left[ \frac{\frac{1}{2}(I_{p-2}\vec{m} + \vec{u}_{r,s})_i}{m_i} \right]_q \quad (4.27)$$

(Notice that we are actually allowed to replace  $\frac{1}{2}\vec{A}_{r,s}^{(p-3)}\vec{m} - \frac{1}{2}\vec{u}_{r,s}^{(p-3)}\vec{m}$  in the exponent by  $\frac{1}{2}\vec{A}_{r,s}\vec{m} - \frac{1}{2}\vec{u}_{r,s}\vec{m}$  since  $\vec{A}_{r,s}^{(p-3)} - \vec{u}_{r,s}^{(p-3)} = -\vec{e}_{p-r-1}$  and  $1 \leq r \leq p-2$  and hence  $\vec{e}_0$  can never be reached). Defining  $\vec{n} = (n_1, n_2, \dots, n_{p-2}) \equiv \vec{m}$  we may simplify (4.27) further to

$$\chi_{s,r}^{(p,p+1)} = q^{-\frac{1}{4}(r-s)(r-s-1)} \sum_{\vec{n} \equiv \vec{Q}_{s,r}^{(p-2)}} \frac{1}{(q)_{n_1}} q^{\frac{1}{4}\vec{n}C_{p-2}\vec{n} - \frac{1}{2}\vec{A}_{s,r}^{(p-2)}\vec{n}} \prod_{i=2}^{p-2} \left[ \frac{\frac{1}{2}(I_{p-2}\vec{n} + \vec{u}_{s,r}^{(p-2)})_i}{n_i} \right]_q \quad (4.28)$$

where

$$\begin{aligned}
\vec{A}_{s,r}^{(p-2)} &= \vec{e}_{p-r} \\
\vec{u}_{s,r}^{(p-2)} &= \vec{e}_s + \vec{e}_{p-r} \\
\vec{Q}_{s,r}^{(p-2)} &= (r-1)\vec{\rho}^{(p-2)} + (\vec{e}_{s-1} + \vec{e}_{s-3} + \dots) + (\vec{e}_{p+1-r} + \vec{e}_{p+3-r} + \dots).
\end{aligned} \tag{4.29}$$

Hence the right hand side of (4.28) is  $F_{s,r}^{(\infty,p+1)}(q)$  with  $\vec{A}_{r,s}^{(p-2)}$ ,  $\vec{u}_{r,s}^{(p-2)}$  and  $\vec{Q}_{r,s}^{(p-2)}$  as in (4.16) (notice that  $r$  and  $s$  are interchanged in  $F_{s,r}^{(\infty,p+1)}$  relative to the definition of  $F_{r,s}^{(L,p)}$  in (4.13)).

Similarly one may show that from (4.23)

$$\chi_{s,r+2}^{(p,p+1)} = F_{s,r+2}^{(\infty,p+1)}(q) \tag{4.30}$$

with  $\vec{A}_{r,s}^{(p-2)}$ ,  $\vec{u}_{r,s}^{(p-2)}$  and  $\vec{Q}_{r,s}^{(p-2)}$  as in (4.15).

Thus all characters of the model  $M(p, p+1)$  have been obtained from the characters of  $M(p-1, p)$  by means of the Bailey construction and hence we have extended the results of [20]. This implementation of the renormalization group flow of [22]-[23] and is what we call Bailey renormalization flow.

#### 4.3. The $N = 1$ supersymmetric model $SM(p, p+2)$

The second specialization of (4.19) is

$$\rho_1 \rightarrow \infty, \quad \rho_2 = -q^{\frac{r-s+1}{2}}. \tag{4.31}$$

Then if in addition we set  $x = 0$  so that  $a = q^{r-s}$  we see that  $\rho_2 = aq/\rho_2$  and all the denominators in (4.19) cancel. Thus we obtain from (4.19) the Neveu-Schwarz characters when  $r-s$  is even

$$\hat{\chi}_{r,s}^{(p+2,p)} = \hat{\chi}_{s,r}^{(p,p+2)} = \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}})_{n+\frac{r-s}{2}}}{(q)_{2n+r-s}} q^{\frac{3}{2}n^2} q^{n\frac{3(r-s)}{2}} F_{r,s}^{(2n+r-s,p)}(q^{-1}) \tag{4.32}$$

and the Ramond characters when  $r-s$  is odd

$$\hat{\chi}_{r,s}^{(p+2,p)} = \hat{\chi}_{s,r}^{(p,p+2)} = \sum_{n=0}^{\infty} \frac{(-q)_{n+\frac{r-s-1}{2}}}{(q)_{2n+r-s}} q^{\frac{3}{2}n^2} q^{n\frac{3(r-s)}{2}} F_{r,s}^{(2n+r-s,p)}(q^{-1}) \tag{4.33}$$

Again it should be mentioned that from the  $M(p-1, p)$  construction  $r, s$  are restricted by  $1 \leq r \leq p-2$ ,  $1 \leq s \leq p-1$  whereas the range for  $r, s$  in  $\hat{\chi}_{s,r}^{(p,p+2)}$  is

$$1 \leq r \leq p+1, \quad 1 \leq s \leq p-1. \quad (4.34)$$

Employing the Bailey pair (4.12) in the identical fashion with the specialization

$$\rho_1 \rightarrow \infty, \quad \rho_2 = -q^{\frac{r-s+2}{2}}, \quad a = q^{r-s+1} \quad (4.35)$$

leads, for the Neveu-Schwarz case when  $r-s+3$  even, to

$$\hat{\chi}_{s,r+3}^{(p,p+2)} = \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}})_{n+\frac{r-s+1}{2}}}{(q)_{2n+r-s+1}} q^{\frac{3}{2}n(n+r-s+1)} F_{r,s}^{(2n+r-s+1,p)}(q^{-1}) \quad (4.36)$$

and for the Ramond case, when  $r-s+3$  odd, to

$$\hat{\chi}_{s,r+3}^{(p,p+2)} = \sum_{n=0}^{\infty} \frac{(-q)_{n+\frac{r-s}{2}}}{(q)_{2n+r-s+1}} q^{\frac{3}{2}n(n+r-s+1)} F_{r,s}^{(2n+r-s+1,p)}(q^{-1}). \quad (4.37)$$

Hence we obtain the fermionic expressions for the whole range in  $r, s$  (4.34) except for the character  $\hat{\chi}_{s,3}^{(4,6)}$ .

Again we compare our results (4.32), (4.33), (4.36), (4.37) with the known results for the fermionic characters for the  $SM(p, p+2)$  model as given in [34] [21] (the fermionic expressions in [21] are actually expressions for the branching functions and to obtain the characters one needs to sum over the index  $l$ )

$$\begin{aligned} \hat{\chi}_{r,s}^{(p,p+2)} &= \epsilon_{r-s} q^{-\frac{1}{8}(s-r-2\epsilon_{r-s}+1)(s-r+2\epsilon_{r-s}-3)} \\ &\times \sum_{m_1 \geq 0} \sum_{m_i \equiv (\vec{Q}_{r,s}^{(p-1)})_i, 2 \leq i \leq p-1} q^{\frac{1}{4}\vec{m}C_{p-1}\vec{m} - \frac{1}{2}\vec{A}_{r,s}^{(p-1)}\vec{m}} \frac{1}{(q)_{m_2}} \prod_{i=1, i \neq 2}^{p-1} \left[ \frac{\frac{1}{2}(I_{p-1}\vec{m} + \vec{u}_{r,s}^{(p-1)})_i}{m_i} \right]_q \end{aligned} \quad (4.38)$$

with  $\epsilon_{r-s}$  as in (2.14),  $C_{p-1}$  the  $(p-1)$  dimensional Cartan matrix of  $A_{p-1}$ ,  $I_{p-1} = 2 - C_{p-1}$  and

$$\begin{aligned} \vec{A}_{r,s}^{(p-1)} &= \vec{e}_{s-1} \\ \vec{u}_{r,s}^{(p-1)} &= \vec{e}_{s-1} + \vec{e}_{p+1-r} + (2\epsilon_{r-s} - 1)\vec{e}_1 \\ \vec{Q}_{r,s}^{(p-1)} &= (r-1)\vec{\rho}^{(p-1)} + (\vec{e}_{s-2} + \vec{e}_{s-4} + \dots) + (\vec{e}_{p+2-r} + \vec{e}_{p+4-r} + \dots) \end{aligned} \quad (4.39)$$



or

$$\begin{aligned}
\vec{A}_{r,s}^{(p-1)} &= \vec{e}_{p+1-s} \\
\vec{u}_{r,s}^{(p-1)} &= \vec{e}_{p+1-s} + \vec{e}_{r+1} + (2\epsilon_{r-s} - 1)\vec{e}_1 \\
\vec{Q}_{r,s}^{(p-1)} &= (s-1)\vec{\rho}^{(p-1)} + (\vec{e}_{p-s+2} + \vec{e}_{p-s+4} + \dots) + (\vec{e}_r + \vec{e}_{r-2} + \dots)
\end{aligned} \tag{4.40}$$

where  $\vec{\rho}^{(p-1)} = \vec{e}_1 + \dots + \vec{e}_{p-1}$ . These two choices for  $\vec{A}_{r,s}^{(p-1)}$ ,  $\vec{u}_{r,s}^{(p-1)}$ ,  $\vec{Q}_{r,s}^{(p-1)}$  reflect the symmetry  $\hat{\chi}_{r,s}^{(p,p+2)} = \hat{\chi}_{p-r,p+2-s}^{(p,p+2)}$ .

To show that (4.32) is of the form (4.38) let us set  $m_0 = 2n + r - s$  and use (4.25) and

$$(-q^{\frac{1}{2}})^{\frac{m_0}{2}} = \sum_{k=0}^{\frac{m_0}{2}} q^{\frac{1}{2}(\frac{m_0}{2}-k)^2} \left[ \frac{\frac{m_0}{2}}{k} \right]_q \tag{4.41}$$

which follows from (2.21). Then we obtain from (4.32)

$$\begin{aligned}
\hat{\chi}_{s,r}^{p,p+2} &= q^{-\frac{1}{8}(r-s)(r-s-2)} \sum_{m_0 \geq 0, \text{even}}^{\infty} \sum_{k=0}^{\frac{m_0}{2}} \sum_{\vec{m} \equiv \vec{Q}_{r,s}^{(p-3)}} q^{\frac{1}{2}m_0^2 - \frac{1}{2}m_0k + \frac{1}{2}k^2 - \frac{1}{2}m_0m_1} \\
&\quad \times q^{\frac{1}{4}\vec{m}C_{p-3}\vec{m} + \frac{1}{2}\vec{A}_{r,s}^{(p-3)}\vec{m} - \frac{1}{2}\vec{u}_{r,s}^{(p-3)}\vec{m}} \left[ \frac{\frac{m_0}{2}}{k} \right]_q \frac{1}{(q)_{m_0}} \prod_{i=1}^{p-3} \left[ \frac{\frac{1}{2}(I_{p-3}\vec{m} + \vec{u}_{r,s}^{(p-3)} + m_0\vec{e}_1)_i}{m_i} \right]_q
\end{aligned} \tag{4.42}$$

where  $\vec{A}_{r,s}^{(p-3)}$ ,  $\vec{u}_{r,s}^{(p-3)}$  and  $\vec{Q}_{r,s}^{(p-3)}$  as given in (4.15). Let us now define  $\vec{n} \equiv (n_1, \dots, n_{p-1}) = (k, m_0, \vec{m})$ ,  $C_{p-1}$  as the Cartan matrix of the Lie algebra  $A_{p-1}$ , and  $I_{p-1} = 2 - C_{p-1}$ . Then we may rewrite (4.42) as

$$\begin{aligned}
\hat{\chi}_{s,r}^{p,p+2} &= q^{-\frac{1}{8}(r-s)(r-s-2)} \sum_{n_1=0}^{\infty} \sum_{n_i \equiv (\vec{Q}_{r,s}^{(p-1)})_i, 2 \leq i \leq p-1} q^{\frac{1}{4}\vec{n}C_{p-1}\vec{n} + \frac{1}{2}\vec{A}_{r,s}^{(p-1)}\vec{n} - \frac{1}{2}\vec{u}_{r,s}^{(p-1)}\vec{n}} \\
&\quad \times \left[ \frac{n_2}{n_1} \right]_q \frac{1}{(q)_{n_2}} \prod_{i=3}^{p-1} \left[ \frac{\frac{1}{2}(I_{p-1}\vec{n} + \vec{u}_{r,s}^{(p-1)})_i}{n_i} \right]_q
\end{aligned} \tag{4.43}$$

where

$$\begin{aligned}
\vec{A}_{r,s}^{(p-1)} &= \vec{e}_{s+1} \\
\vec{u}_{r,s}^{(p-1)} &= \vec{e}_{s+1} + \vec{e}_{p+1-r} \\
\vec{Q}_{r,s}^{(p-1)} &= (r-1)\vec{\rho}^{(p-1)} + (\vec{e}_s + \vec{e}_{s-2} + \dots) + (\vec{e}_{p-r+2} + \vec{e}_{p-r+4} + \dots)
\end{aligned} \tag{4.44}$$

with  $(\vec{e}_i)_j = \delta_{ij}$  for  $1 \leq i \leq p-1$  and  $\vec{\rho}^{(p-1)} = \vec{e}_1 + \dots + \vec{e}_{p-1}$ . Again, we are allowed to replace  $\vec{A}_{r,s}^{(p-3)}\vec{m} - \vec{u}_{r,s}^{(p-3)}\vec{m}$  by  $\vec{A}_{r,s}^{(p-1)}\vec{n} - \vec{u}_{r,s}^{(p-1)}\vec{n}$  because  $\vec{A}_{r,s}^{(p-1)} - \vec{u}_{r,s}^{(p-1)} = -\vec{e}_{p+1-r}$

and  $1 \leq r \leq p-2$ . Hence (4.43) agrees with  $\hat{\chi}_{s,r}^{(p,p+2)}$  as in (4.38) with  $\vec{A}_{r,s}^{(p-1)}$ ,  $\vec{u}_{r,s}^{(p-1)}$  and  $\vec{Q}_{r,s}^{(p-1)}$  as in (4.40) in the Neveu–Schwarz sector.

Similarly we may treat (4.33) by using

$$(-q)^{\frac{m_0-1}{2}} = \frac{1}{2}(-1)^{\frac{m_0+1}{2}} = \frac{1}{2} \sum_{k=0}^{\frac{m_0+1}{2}} q^{\frac{1}{2}(\frac{m_0+1}{2}-k)(\frac{m_0-1}{2}-k)} \begin{bmatrix} \frac{m_0+1}{2} \\ k \end{bmatrix}_q \quad (4.45)$$

and we find the Ramond character

$$\begin{aligned} \hat{\chi}_{s,r}^{(p,p+2)} &= \frac{1}{2} q^{-\frac{1}{8}(r-s-1)^2} \sum_{n_1=0}^{\infty} \sum_{n_i \equiv (\vec{Q}_{r,s}^{(p-1)})_i, 2 \leq i \leq p-1} q^{\frac{1}{4}\vec{n}C_{p-1}\vec{n} + \frac{1}{2}\vec{A}_{r,s}^{(p-1)}\vec{n} - \frac{1}{2}\vec{u}_{r,s}^{(p-1)}\vec{n}} \\ &\times \begin{bmatrix} \frac{n_2+1}{2} \\ n_1 \end{bmatrix}_q \frac{1}{(q)_{n_2}} \prod_{i=3}^{p-1} \begin{bmatrix} \frac{1}{2}(I_{p-1}\vec{n} + \vec{u}_{r,s}^{(p-1)})_i \\ n_i \end{bmatrix}_q \end{aligned} \quad (4.46)$$

where  $\vec{A}_{r,s}^{(p-1)}$ ,  $\vec{u}_{r,s}^{(p-1)}$ ,  $\vec{Q}_{r,s}^{(p-1)}$  as in (4.44). Hence (4.46) also agrees with  $\hat{\chi}_{s,r}^{(p,p+2)}$  as in (4.38) with  $\vec{A}_{r,s}^{(p-1)}$ ,  $\vec{u}_{r,s}^{(p-1)}$ ,  $\vec{Q}_{r,s}^{(p-1)}$  as in (4.40).

Analogously one finds that (4.36) ((4.37)) where  $\vec{A}_{r,s}^{(p-3)}$ ,  $\vec{u}_{r,s}^{(p-3)}$ ,  $\vec{Q}_{r,s}^{(p-3)}$  given by (4.16) can be written in the form (4.38) with  $\vec{A}_{r,s}^{(p-1)}$ ,  $\vec{u}_{r,s}^{(p-1)}$ ,  $\vec{Q}_{r,s}^{(p-1)}$  as in (4.39).

#### 4.4. $N = 2$ characters with $c = 3(1 - 2/p)$

Finally we consider the case when

$$\rho_1 \text{ finite}, \quad \rho_2 \text{ finite}. \quad (4.47)$$

Again we distinguish the three sectors  $A$ ,  $P$  and  $T$ . We start with sector  $A$  and set  $r = 1$  and  $\tilde{a} = \frac{aq}{\rho_1\rho_2}$  in (4.19). As in section 2.3 we are interested in the limit  $\tilde{a} \rightarrow 1$ . With these definitions the left hand side of (4.19) becomes

$$\begin{aligned} &q^{x(s-1-x)} \frac{(\tilde{a}\rho_1)_{\infty}(\tilde{a}\rho_2)_{\infty}}{(\tilde{a}\rho_1\rho_2)_{\infty}(\tilde{a})_{\infty}} \sum_{j=-\infty}^{\infty} \frac{(\rho_1)_{pj-x-1}(\rho_2)_{pj-x-1}}{(\tilde{a}\rho_1)_{pj-x-1}(\tilde{a}\rho_2)_{pj-x-1}} \tilde{a}^{pj-x-1} q^{j^2p-sj} \\ &\times \left( \frac{(1-\rho_1q^{pj-x-1})(1-\rho_2q^{pj-x-1})}{(1-\tilde{a}\rho_1q^{pj-x-1})(1-\tilde{a}\rho_2q^{pj-x-1})} \tilde{a} - 1 \right) \\ &= q^{x(s-1-x)} \frac{(\tilde{a}\rho_1)_{\infty}(\tilde{a}\rho_2)_{\infty}}{(\tilde{a}\rho_1\rho_2)_{\infty}(\tilde{a})_{\infty}} \sum_{j=-\infty}^{\infty} \frac{(\rho_1)_{pj-x-1}(\rho_2)_{pj-x-1}}{(\tilde{a}\rho_1)_{pj-x-1}(\tilde{a}\rho_2)_{pj-x-1}} \tilde{a}^{pj-x-1} q^{j^2p-sj} \\ &\times (1-\tilde{a}) \frac{\tilde{a}\rho_1\rho_2q^{2pj-2x-2} - 1}{(1-\tilde{a}\rho_1q^{pj-x-1})(1-\tilde{a}\rho_2q^{pj-x-1})} \\ &\rightarrow_{\tilde{a} \rightarrow 1} q^{x(s-1-x)} \frac{(\rho_1)_{\infty}(\rho_2)_{\infty}}{(\rho_1\rho_2)_{\infty}(q)_{\infty}} \sum_{j=-\infty}^{\infty} q^{j^2p-sj} \frac{\rho_1\rho_2q^{2pj-2x-2} - 1}{(1-\rho_1q^{pj-x-1})(1-\rho_2q^{pj-x-1})}. \end{aligned} \quad (4.48)$$

Let us set

$$\rho_1 = -yq^{\frac{1}{2}}, \quad \rho_2 = -y^{-1}q^{\frac{1}{2}}q^{\hat{r}-\hat{s}} \quad (4.49)$$

where  $\hat{r}$  and  $\hat{s}$  are half-integers such that  $\hat{r} \geq \hat{s}$  and  $s = \hat{r} + \hat{s}$ . From this follows that  $x = \hat{r} - \frac{1}{2}$  since  $\frac{aq}{\rho_1\rho_2} = 1$ . Then changing  $j \rightarrow -j$  we obtain from (4.48) and (4.19)

$$\begin{aligned} & \frac{(-yq^{\frac{1}{2}})_\infty (-y^{-1}q^{\frac{1}{2}})_\infty}{(q)_\infty^2} \sum_{j=-\infty}^{\infty} q^{j^2 p + (\hat{r}+\hat{s})j} \frac{1 - q^{2pj+\hat{r}+\hat{s}}}{(1 + y^{-1}q^{pj+\hat{r}})(1 + yq^{pj+\hat{s}})} \\ &= q^{-(\hat{r}-\frac{1}{2})(\hat{s}-\frac{1}{2})} \sum_{n=0}^{\infty} \frac{(-yq^{\frac{1}{2}})_n (-y^{-1}q^{\frac{1}{2}})_n}{(q)_{2n+\hat{r}-\hat{s}}} q^{n(n+\hat{r}-\hat{s})} F_{1,\hat{r}+\hat{s}}^{(2n+\hat{r}-\hat{s},p)}(q^{-1}) \end{aligned} \quad (4.50)$$

which agrees with the  $N = 2$  unitary characters (2.29)  $\chi_{\hat{r},\hat{s}}^{(N=2)(p)}$  with central charge  $c = 3(1 - 2/p)$ .

For the  $P$  sector we set

$$\rho_1 = -yq, \quad \rho_2 = -y^{-1}qq^{\hat{r}-\hat{s}} \quad (4.51)$$

where now  $\hat{r}, \hat{s}$  are integers,  $\hat{r} \geq \hat{s}$  and  $s = \hat{r} + \hat{s}$ . From  $\frac{aq}{\rho_1\rho_2} = 1$  follows that  $x = \hat{r}$ . Hence we obtain

$$\begin{aligned} & \frac{(-yq)_\infty (-y^{-1}q)_\infty}{(q)_\infty^2} \sum_{j=-\infty}^{\infty} q^{j^2 p + (\hat{r}+\hat{s})j} \frac{1 - q^{2pj+\hat{r}+\hat{s}}}{(1 + y^{-1}q^{pj+\hat{r}})(1 + yq^{pj+\hat{s}})} \\ &= q^{-\hat{r}(\hat{s}-1)} \sum_{n=0}^{\infty} \frac{(-yq)_n (-y^{-1}q)_n}{(q)_{2n+\hat{r}-\hat{s}+1}} q^{n(n+\hat{r}-\hat{s}+1)} F_{1,\hat{r}+\hat{s}}^{(2n+\hat{r}-\hat{s}+1,p)}(q^{-1}) \end{aligned} \quad (4.52)$$

which is again in agreement with (2.29) in the  $P$  sector.

Finally we need to consider the  $T$  sector. Here we set  $r = 1$  and when  $s$  odd

$$\rho_1 = -q^{\frac{1}{2}}, \quad \rho_2 = -q, \quad a = 1. \quad (4.53)$$

Since  $a = q^{r-s+2x}$  we read off  $x = \frac{s-1}{2}$ . Inserting this into the left hand side of (4.19) yields

$$\begin{aligned} & \frac{(-q^{\frac{1}{2}})_\infty (-q)_\infty}{(q)_\infty (q^{-\frac{1}{2}})_\infty} \sum_{j=-\infty}^{\infty} q^{j(jp-s)} q^{x(s-1-x)} \\ & \quad \times \left( (1 + q^{pj-x}) q^{-\frac{1}{2}(pj-x)} - (1 + q^{pj-x-1}) q^{-\frac{1}{2}(pj-x-1)} \right) \\ &= \frac{(-q^{\frac{1}{2}})_\infty (-q)_\infty}{(q)_\infty (q^{-\frac{1}{2}})_\infty} \sum_{j=-\infty}^{\infty} q^{j(jp-s)} q^{\frac{1}{4}(s-1)^2} \left( q^{-\frac{1}{2}(pj-x)} (1 - q^{-\frac{1}{2}}) (-q^{\frac{1}{2}} + q^{pj-x}) \right) \\ &= \frac{(-q^{\frac{1}{2}})_\infty (-q)_\infty}{(q)_\infty (q^{\frac{1}{2}})_\infty} q^{\frac{1}{4}(s-1)(s-2)} \sum_{j=-\infty}^{\infty} \left( q^{j^2 p - js + \frac{1}{2}pj} - q^{j^2 p - js - \frac{1}{2}pj + \frac{s}{2}} \right). \end{aligned} \quad (4.54)$$

Hence we obtain

$$\chi_s^{(N=2)(p)} = q^{-\frac{1}{4}(s-1)(s-2)} \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}})_n (-q)_n}{(q)_{2n}} q^{n(n-\frac{1}{2})} F_{1,s}^{(2n,p)}(q^{-1}) \quad (4.55)$$

where  $\chi_s^{(N=2,T)(p)}$  is the character given in (2.32).

For  $s$  even we choose

$$\rho_1 = -q^{\frac{3}{2}}, \quad \rho_2 = -q, \quad a = q \quad (4.56)$$

which yields  $x = \frac{s}{2}$ . A similar calculation gives

$$\begin{aligned} \chi_s^{(N=2)(p)} &= \frac{(-q^{\frac{1}{2}})_{\infty} (-q)_{\infty}}{(q)_{\infty} (q^{\frac{1}{2}})_{\infty}} \sum_{j=-\infty}^{\infty} \left( q^{j^2 p - js + \frac{1}{2} jp} - q^{j^2 p - js - \frac{1}{2} jp + \frac{s}{2}} \right) \\ &= q^{-\frac{1}{4}(s-1)(s-2)} \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}})_{n+1} (-q)_n}{(q)_{2n+1}} q^{n^2 + \frac{n}{2}} F_{1,s}^{(2n+1,p)}(q^{-1}). \end{aligned} \quad (4.57)$$

To calculate the explicit fermionic form in the different sectors let us first introduce the following matrix

$$C_{D_p} = \begin{pmatrix} 2 & 0 & -1 & 0 & \dots \\ 0 & 2 & -1 & 0 & \dots \\ -1 & -1 & & & \\ 0 & 0 & & C_{p-2} & \\ \vdots & \vdots & & & \end{pmatrix} \quad (4.58)$$

where  $C_{p-2}$  is the  $(p-2)$  dimensional Cartan matrix of  $A_{p-2}$ .  $C_{D_p}$  is the Cartan matrix of the Lie algebra  $D_p$ .

Let us start with the  $A$  sector and define  $m_0 = 2n + \hat{r} - \hat{s}$ . Then (4.50) becomes

$$\begin{aligned} \chi_{\hat{r}, \hat{s}}^{(N=2)(p)} &= q^{-(\hat{r}-\frac{1}{2})(\hat{s}-\frac{1}{2})} \sum_{m_0=0, \text{restr.}}^{\infty} \frac{1}{(q)_{m_0}} q^{\frac{1}{4}(m_0-\hat{r}+\hat{s})(m_0+\hat{r}-\hat{s})} F_{1, \hat{r}+\hat{s}}^{(m_0,p)}(q^{-1}) \\ &\times \left( \sum_{k_1=0}^{\infty} y^{\frac{m_0-\hat{r}+\hat{s}}{2}-k_1} q^{\frac{1}{2}(\frac{m_0-\hat{r}+\hat{s}}{2}-k_1)^2} \begin{bmatrix} \frac{m_0-\hat{r}+\hat{s}}{2} \\ k_1 \end{bmatrix}_q \right) \\ &\times \left( \sum_{k_2=0}^{\infty} y^{-(\frac{m_0+\hat{r}-\hat{s}}{2}-k_2)} q^{\frac{1}{2}(\frac{m_0+\hat{r}-\hat{s}}{2}-k_2)^2} \begin{bmatrix} \frac{m_0+\hat{r}-\hat{s}}{2} \\ k_2 \end{bmatrix}_q \right) \end{aligned} \quad (4.59)$$

where we used (2.21) twice to convert  $(-yq^{\frac{1}{2}})_n$  and  $(-y^{-1}q^{\frac{1}{2}})_{n+\hat{r}-\hat{s}}$  into the  $k_1$  and  $k_2$  sums, respectively. The restriction on  $m_0$  is such that  $m_0$  is even if  $\hat{r} + \hat{s}$  odd and vice

versa (remember that  $\hat{r}, \hat{s}$  are half-integers in the  $A$  sector). Then using (4.25) and setting  $\vec{n} = (k_1, k_2, m_0, \vec{m})$  we finally obtain

$$\begin{aligned} \chi_{\hat{r}, \hat{s}}^{(N=2)(p)} &= q^{\frac{1}{4}(\hat{r}^2 + \hat{s}^2 - 2\hat{r}\hat{s} - \hat{r} - \hat{s} + 1)} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_i \equiv (\vec{Q}_{\hat{r}, \hat{s}}^{(p)})_i, 3 \leq i \leq p} y^{\hat{s} - \hat{r} + n_2 - n_1} \\ &\times q^{\frac{1}{4}\vec{n}C_{D_p}\vec{n} + \frac{1}{2}(n_1 - n_2)(\hat{r} - \hat{s})} \frac{1}{(q)_{n_3}} \prod_{i=1, i \neq 3}^p \left[ \begin{matrix} \frac{1}{2}(I_{D_p}\vec{n} + \vec{u}_{\hat{r}, \hat{s}}^{(p)})_i \\ n_i \end{matrix} \right]_q \end{aligned} \quad (4.60)$$

where

$$\begin{aligned} \vec{Q}_{\hat{r}, \hat{s}}^{(p)} &= \vec{e}_{\hat{r} + \hat{s} + 1} + \vec{e}_{\hat{r} + \hat{s} - 1} + \dots \\ \vec{u}_{\hat{r}, \hat{s}}^{(p)} &= \vec{e}_{\hat{r} + \hat{s} + 2} + (\hat{s} - \hat{r})\vec{e}_1 + (\hat{r} - \hat{s})\vec{e}_2. \end{aligned} \quad (4.61)$$

Here  $\vec{e}_i$  are the  $p$  dimensional unit vectors in direction  $i$  for  $1 \leq i \leq p$ .

Similarly we get the fermionic character expression for the  $P$  sector from (4.52) with  $m_0 = 2n + \hat{r} - \hat{s} + 1$  and  $\vec{n} = (k_1, k_2, m_0, \vec{m})$

$$\begin{aligned} \chi_{\hat{r}, \hat{s}}^{(N=2)(p)} &= q^{\frac{1}{4}(\hat{r}^2 + \hat{s}^2 - 2\hat{r}\hat{s} - \hat{r} - \hat{s})} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_i \equiv (\vec{Q}_{\hat{r}, \hat{s}}^{(p)})_i, 3 \leq i \leq p} y^{\hat{s} - \hat{r} + n_2 - n_1} \\ &\times q^{\frac{1}{4}\vec{n}C_{D_p}\vec{n} + \frac{1}{2}(n_1 - n_2)(\hat{r} - \hat{s})} \frac{1}{(q)_{n_3}} \prod_{i=1, i \neq 3}^p \left[ \begin{matrix} \frac{1}{2}(I_{D_p}\vec{n} + \vec{u}_{\hat{r}, \hat{s}}^{(p)})_i \\ n_i \end{matrix} \right]_q \end{aligned} \quad (4.62)$$

where

$$\begin{aligned} \vec{Q}_{\hat{r}, \hat{s}}^{(p)} &= \vec{e}_{\hat{r} + \hat{s} + 1} + \vec{e}_{\hat{r} + \hat{s} - 1} + \dots \\ \vec{u}_{\hat{r}, \hat{s}}^{(p)} &= \vec{e}_{\hat{r} + \hat{s} + 2} + (\hat{s} - \hat{r} - 1)\vec{e}_1 + (\hat{r} - \hat{s} - 1)\vec{e}_2. \end{aligned} \quad (4.63)$$

For the  $T$  sector we get from (4.55) and (4.57) using (2.21) and (4.25)

$$\chi_s^{(N=2)(p)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_i \equiv (\vec{Q}_s^{(p)})_i, 3 \leq i \leq p} q^{\frac{1}{4}\vec{n}C_{D_p}\vec{n} - \frac{1}{2}n_1} \frac{1}{(q)_{n_3}} \prod_{i=1, i \neq 3}^p \left[ \begin{matrix} \frac{1}{2}(I_{D_p}\vec{n} + \vec{u}_s^{(p)})_i \\ n_i \end{matrix} \right]_q \quad (4.64)$$

where

$$\begin{aligned} \vec{Q}_s^{(p)} &= \vec{e}_{s+1} + \vec{e}_{s-1} + \dots \\ \vec{u}_s^{(p)} &= \vec{e}_{s+2} + \begin{cases} \vec{e}_1 - \vec{e}_2 & \text{for } s \text{ even} \\ \vec{0} & \text{for } s \text{ odd} \end{cases}. \end{aligned} \quad (4.65)$$

## 5. Conclusion

We have now demonstrated in detail that the construction of Bailey when applied to the minimal model  $M(p, p+1)$  leads to the  $N = 1$  supersymmetric model  $SM(p, p+2)$  and the  $N = 2$  unitary supersymmetric model with  $c = 3(1-2/p)$ . However, this construction is merely illustrative and the method is of great generality. In particular we can show [25] that if we start from the dual Bailey pair constructed from the general model  $M(p, p')$  (with  $p < p'$ ) we obtain the minimal models  $M(p', 2p' - p)$ ,  $SM(p', 3p' - 2p)$  and the nonunitary  $N = 2$  models with  $c = 3(2p - p')/p'$  while from the direct Bailey pair of  $M(p, p')$  we have the sequence  $M(p', p' + p)$ ,  $SM(p', p' + 2p)$  and  $N = 2$  with  $c = -3(2p - p')/p'$ . Our results are obtained from the general bosonic form of Forrester and Baxter [35] and the fermionic results of [24].

In the unitary case presented in this paper the fermionic expressions (4.60), (4.62) and (4.64) for the characters in terms of the Cartan matrix of the group  $D_p$  may be interpreted in terms of the construction of Zamolodchikov and Fateev [36] in terms of parafermions and two Majorana fermions if we identify the two variables on the forks of the Dynkin diagram with the Majorana fermions and the rest of the diagram with the parafermions  $M(1, p+1)$  (which are dual to  $M(p, p+1)$ .)

The general nonunitary case the  $N = 2$  are to be compared with the string theory results of [1]–[3] and the flows of minimal models  $M(p, p')$  to  $M(p', 2p' - p)$  is that of [37]. However, the identification of these character expressions in the general nonunitary case is more cumbersome than what was done here in the unitary case and will be treated separately [25]. The flow  $M(p, p')$  to  $M(p', p' + p)$  does not seem to have been previously seen.

We also note that the existing computations for the quantum gravity models are only made explicit for the  $W_2$  and  $W_3$  gravities. The  $W_2$  case is what is treated here and the  $W_3$  case should correspond to the Bailey pairs of Milne and Lilly [13] constructed from  $SU(3)$ . However, in [13] the Bailey lemma is derived for all the Lie groups  $A_n$  and  $C_n$  and this should correspond to results for all the  $W_n$  gravities.

But probably the most provocative question is to find an interpretation not only for the Bailey formula for the specialized values of  $\rho_1$  and  $\rho_2$  but for general values of these parameters. These parameters seem to be playing the role of fugacities and the Bailey pairs seem to be building up complex systems by gluing these more elementary fermions together. Thus the parameters would seem to govern a renormalization flow between models. All of these questions deserve further study.

### **Acknowledgements**

The authors are pleased to acknowledge many useful conversations with G.Andrews, J. de Boer, V. Dobrev, O. Foda, W. Nahm, M.Roesgen, S.O. Warnaar, and N. Warner. This work is supported in part by the NSF under DMR9404747.

## References

- [1] B. Gato–Rivera and A.M. Semikhatov, Phys. Lett. B293 (1992) 72
- [2] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Phys. Lett. B292 (1992) 35.
- [3] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Nucl. Phys. B401 (1993) 304.
- [4] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193
- [5] E. Date, M. Jimbo, T. Miwa, and M. Okado, Phys. Rev. B 35 (1987) 2105.
- [6] W.N. Bailey, Proc. Lond. Math. Soc (2) 50 (1949) 1.
- [7] L.J. Rogers, Proc. Lond. Math. Soc. 25 (1894) 318.
- [8] L.J. Rogers, Proc. Lond. Math. Soc. (2) 16 (1917) 315.
- [9] L.J. Slater, Proc. Lond. Math. Soc. (2) 52 (1951) 460.
- [10] L.J. Slater, Proc. Lond. Math. Soc. (2) 54 (1951-52) 147.
- [11] G.E. Andrews, Pac. J. Math. 114,(1984) 276.
- [12] A.K. Agarwal, G.E. Andrews and D.M. Bressoud, J. Indian Math. Soc. 51 (1987) 57.
- [13] S.C. Milne and G. M. Lilly, Bull. Am. Math. Soc. (NS) 26 (1992) 258.
- [14] O. Foda and Y–H. Quano, hep-th 9408086.
- [15] E. Melzer, Int. J. Mod. Phys. A 9 (1994) 1115.
- [16] A. Berkovich, Nucl. Phys. B431 (1994) 315.
- [17] O. Foda and Y–H. Quano, hep-th 9407191.
- [18] R. Kedem, T.R Klassen, B.M. McCoy and E. Melzer, Phys. Letts. B 307 (1993) 68.
- [19] S.O. Warnaar, J. Stat. Phys. (in press) hep-th 9501134.
- [20] S.O. Warnaar, J. Stat. Phys. (in press) hep-th 9508079.
- [21] A. Schilling, Nucl. Phys. B (in press) hep-th 9508050 and 9510168.
- [22] Al. Zamolodchikov, Nucl. Phys. B358 (1991) 524.
- [23] G. Feverati, E. Quattrini and F. Ravinini, hep-th 9412104.
- [24] A. Berkovich and B.M. McCoy, Letts. Math. Phys. (in press) hep-th 9412030.
- [25] A. Berkovich, B.M. McCoy and A. Schilling, (in preparation)
- [26] W.N. Bailey, Quart. J. Math. (Oxford), 7 (1936) 105.
- [27] G.E. Andrews, in *The Theory and Application of Special Functions* (R. Askey, ed.) Academic Press, New York (1975) 191.
- [28] A. Rocha-Caridi, in *Vertex Operators in Mathematics and Physics*. ed. J. Lepowsky, S. Mandelstam and I.M. Singer (Springer, Berlin 1985).
- [29] P. Goddard, A. Kent and D. Olive, Comm. Math. Phys. 103 (198) 105.
- [30] Y. Matsuo, Prog. Theo. Phys. 77 (1987) 793.
- [31] V.K. Dobrev, Phys. Letts. B186 (1987) 43.
- [32] F. Ravanini and S–K. Yang, Phys. Letts. B (1987) 202.
- [33] E.B. Kiritsis, Int. J. Mod. Phys. A3 (1988) 1871.



- [34] E. Baver and D. Gepner, hep-th 9502118.
- [35] P.J. Forrester and R.J. Baxter, J. Stat. Phys. 38 (1985) 435.
- [36] A.B. Zamolochikov and V.A. Fateev, Sov. Phys. JETP 63 (1986) 913.
- [37] C. Ahn, Phys. Lett. B 294 (1992) 204.